KPZ EQUATION WITH FRACTIONAL DERIVATIVES OF WHITE NOISE

MASATO HOSHINO

Abstract. In this paper, we consider the KPZ equation driven by space-time white noise replaced with its fractional derivatives of order $\gamma > 0$ in spatial variable. A well-posedness theory for the KPZ equation is established by Hairer [3] as an application of the theory of regularity structures. Our aim is to see to what extent his theory works if noises become rougher. We can expect that his theory works if and only if $\gamma < 1/2$. However, we show that the renormalization like "$(\partial_x h)^2 - \infty$" is well-posed only if $\gamma < 1/4$.

1. Introduction

In this paper, we discuss the stochastic partial differential equation

$$\partial_t h(t,x) = \partial_x^2 h(t,x) + (\partial_x h(t,x))^2 + \partial_x^\gamma \xi(t,x) \tag{1.1}$$

for $(t,x) \in [0,\infty) \times \mathbb{T}$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, which is equivalent to $[0,1]$ with periodic boundary conditions, and $\gamma \geq 0$. Here $h(t,x)$ is a continuous stochastic process and ξ is a space-time white noise. $\partial_x^\gamma = -(-\partial_x^2)^{\frac{\gamma}{2}}$ is the fractional derivative. If $\gamma = 0$, (1.1) is the KPZ equation, which is proposed in [7] as a model of surface growth.

The equation (1.1) is ill-posed. Formally speaking, h has the same regularity as the solution of the linear equation

$$\partial_t h(t,x) = \partial_x^2 h(t,x) + \partial_x^\gamma \xi(t,x). \tag{1.2}$$

Then $h(t,\cdot)$ belongs to Hölder space $\mathcal{C}^\alpha(\mathbb{T})$ with $\alpha < \frac{1}{2} - \gamma$ for each fixed t. However this implies that the nonlinear term $(\partial_x h)^2$ is the square of the distribution, which generally does not make sense.

Hairer discussed the solution of the KPZ equation in [2] and [4]. It is natural to replace ξ by a smooth approximation ξ_ϵ, which is obtained by a convolution with a smooth mollifier, and consider the classical solution of the KPZ equation with ξ_ϵ. He showed that there exists a sequence of constants $C_\epsilon \sim \frac{1}{\epsilon}$ such that, the sequence of solutions h_ϵ of

$$\partial_t h_\epsilon(t,x) = \partial_x^2 h_\epsilon(t,x) + (\partial_x h_\epsilon(t,x))^2 - C_\epsilon + \xi_\epsilon(t,x) \tag{1.3}$$

has a unique limit h in probability, which is independent of the choice of a mollifier.

Our goal is to make the noise rougher and see to what extent this theory works. Because of the "local subcriticality" (Assumption 8.3 of [3]), we can expect that

Date: February 16, 2016.
1991 *Mathematics Subject Classification.* 35J60, 60H15, 60H40.
Key words and phrases. KPZ equation, fractional derivatives of white noise, regularity structures, renormalization.

similar results hold if $\gamma < \frac{1}{2}$. If we write $h^\delta(t,x) = \delta^{-\frac{1}{2}+\gamma} h(\delta^2 t, \delta x)$ and $\xi^\delta(t,x) = \delta^{\frac{3}{2}} \xi(\delta^2 t, \delta x)$ for $\delta > 0$, then ξ^δ is equal to ξ in distribution and h^δ satisfies

$$\partial_t h^\delta(t,x) = \partial_x^2 h^\delta(t,x) + \delta^{\frac{1}{2}-\gamma}(\partial_x h^\delta(t,x))^2 + \partial_x^\gamma \xi^\delta(t,x).$$

As $\delta \to 0$, we can see that the nonlinear term vanishes. Formally speaking, this means that h behaves like the solution of (1.2) at small scales. His theory implies that it is possible to devise a suitable renormalization in this case. However, we prove that the renormalization like (1.3) is possible only if $\gamma < \frac{1}{4}$ in this paper. In the case $\gamma \geq \frac{1}{4}$, see Subsection 4.9.

Theorem 1.1. *Let ρ be a function on $\mathbb{R}^2$ which is smooth, compactly supported, symmetric in x, nonnegative, and satisfies $\int_{\mathbb{R}^2} \rho(t,x) dt dx = 1$. Set $\rho_\epsilon(t,x) = \epsilon^{-3}\rho(\epsilon^{-2}t, \epsilon^{-1}x)$ for $\epsilon > 0$, and $\xi_\epsilon = \xi * \rho_\epsilon$ (space-time convolution). Let $0 \leq \gamma < \frac{1}{4}$ and $0 < \eta < \frac{1}{2} - \gamma$. Then there exists a sequence of constants $C_\epsilon \sim_{\gamma,\rho} \epsilon^{-1-2\gamma}$ such that, for every initial condition $h_0 \in \mathcal{C}^\alpha(\mathbb{T})$ the sequence of solutions h_ϵ of*

$$\partial_t h_\epsilon(t,x) = \partial_x^2 h_\epsilon(t,x) + (\partial_x h_\epsilon(t,x))^2 - C_\epsilon + \partial_x^\gamma \xi_\epsilon(t,x)$$

converges to a unique stochastic process h, which is independent of the choice of ρ. Precisely, $h_\epsilon(t,\cdot)$ exists until the survival time $T_\epsilon \in (0,\infty]$ which satisfies $\liminf_{\epsilon \downarrow 0} T_\epsilon > 0$, and h_ϵ convergences to h in probability in the uniform norm on $[0,T] \times \mathbb{T}$ and η-Hölder norm on all compact sets in $(0,T] \times \mathbb{T}$ for every $T < \liminf_\epsilon T_\epsilon$.

This theorem is obtained from Theorem 3.7, Proposition 4.4 and Theorem 4.5 for $0 \leq \gamma < \frac{1}{6}$, and from Theorem 3.7, Proposition 4.7 and Theorem 4.8 for $\frac{1}{6} \leq \gamma < \frac{1}{4}$. The estimate of C_ϵ is in Propositions 5.2 and 7.2.

We note that above result is related to [5], where the equation

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + u \partial_t \partial_x W(t,x)$$

is studied. Here W is a standard Brownian motion in t and a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ in x. The relation between both equations is $H = \frac{1}{2} - \gamma$, so that the same boundary $\gamma = H = \frac{1}{4}$ appears. Although both equations are same after performing the Cole-Hopf transformation $u = e^h$, their results do not imply Theorem 1.1 since Itô calculus is only stable under spatial regularizations.

The organization of this paper is as follows. In Section 2, we introduce some notations and fractional calculus. In Section 3, we briefly recall the theory of regularity structures and prepare some tools for the proof of Theorem 1.1. We discuss the renormalization of models in Section 4. Details of the proof are given in Section 5 ($0 \leq \gamma < \frac{1}{10}$), Section 6 ($\frac{1}{10} \leq \gamma < \frac{1}{6}$), Section 7 ($\frac{1}{6} \leq \gamma < \frac{3}{14}$), and Section 8 ($\frac{3}{14} \leq \gamma < \frac{1}{4}$).

2. Notations and fractional calculus

We introduce some notations and definitions.

2.1. **Notations.** For functions $A(x)$ and $B(x)$ of a variable x, we write $A(x) \lesssim B(x)$ if there exists a constant $C > 0$ which is independent of x and we have $A(x) \leq CB(x)$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. If we want to emphasize that C depends on another variable y, then we write $A \lesssim_y B$.

We denote by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the 1-dimensional torus. For a function f on $\mathbb{R}$, we write $f \in L^2(\mathbb{T})$ if and only if $f(\cdot + n) = f$ for every $n \in \mathbb{Z}$ and $\|f\|_{L^2(\mathbb{T})} := (\int_0^1 |f(x)|^2 dx)^{\frac{1}{2}} < \infty$. For $f, g \in L^2(\mathbb{T})$, we define

$$(f, g)_{L^2(\mathbb{T})} = \int_0^1 f(x)\overline{g(x)}dx.$$

For a function f on $\mathbb{R}^2$, we write $f \in L^2(\mathbb{R}\times\mathbb{T})$ if and only if $f(t, \cdot) \in L^2(\mathbb{T})$ for every $t \in \mathbb{R}$ and $\|f\|_{L^2(\mathbb{R}\times\mathbb{T})} := (\int_{(t,x)\in\mathbb{R}\times[0,1)} |f(t, x)|^2 dtdx) < \infty$. For $f, g \in L^2(\mathbb{R} \times \mathbb{T})$, we define

$$(f, g)_{L^2(\mathbb{R}\times\mathbb{T})} = \int_{(t,x)\in\mathbb{R}\times[0,1)} f(t, x)\overline{g(t, x)}dtdx.$$

We denote by $\mathcal{F}f = \mathcal{F}_{\mathbb{T}}f$ the Fourier transform of $f \in L^2(\mathbb{T})$, which is defined by

$$\mathcal{F}f(n) = \int_0^1 f(x)e^{-2\pi inx}dx \quad (n \in \mathbb{Z}).$$

For $f \in L^2(\mathbb{R} \times \mathbb{T})$, we define $\mathcal{F}f(t, n) = \mathcal{F}(f(t, \cdot))(n)$.

We denote by $\mathcal{F}f = \mathcal{F}_{\mathbb{R}}f$ the Fourier transform of $f \in L^1(\mathbb{R})$, which is defined by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}dx \quad (\xi \in \mathbb{R}).$$

For a function f on $\mathbb{R}^2$ such that $f(t, \cdot) \in L^1(\mathbb{R})$ for every $t \in \mathbb{R}$, we define $\mathcal{F}f(t, \xi) = \mathcal{F}(f(t, \cdot))(\xi)$.

For a function f on $\mathbb{R}$ decreasing sufficiently fast as $|x| \to \infty$, we define a function πf by

$$(\pi f)(x) = \sum_{n\in\mathbb{Z}} f(x + n) \quad (x \in \mathbb{R}).$$

We note that $\mathcal{F}_{\mathbb{T}}(\pi f)(n) = \mathcal{F}_{\mathbb{R}}f(n)$ for every $n \in \mathbb{Z}$. For a suitable function f on $\mathbb{R}^2$, we write $(\pi f)(t, x) = \pi(f(t, \cdot))(x)$.

For $r \in \mathbb{Z}_+ := \mathbb{Z}\cap[0, \infty)$, we denote by $C^r(\mathbb{R}^2)$ the space of r times continuously differentiable functions on $\mathbb{R}^2$. We also denote by $C_0^r(\mathbb{R}^2)$ the space of compactly supported functions $f \in C^r(\mathbb{R}^2)$. We write $\mathcal{B}_0^r$ for the set of all functions $f \in C_0^r(\mathbb{R}^2)$ such that $\|f\|_{C^r} := \sum_{k_0+k_1\le r} \|\partial_t^{k_0}\partial_x^{k_1} f\|_{L^\infty} \le 1$ and $\operatorname{supp} f \subset \{(t, x) \in \mathbb{R}^2 ; |t|^{\frac{1}{2}} + |x| \le 1\}$.

We denote by $\mathcal{S}(\mathbb{R}^2)$ the space of Schwartz functions on $\mathbb{R}^2$. We denote by $\mathcal{S}'(\mathbb{R}^2)$ its dual. We also denote by $(C_0^r)'(\mathbb{R}^2)$ the dual of $C_0^r(\mathbb{R}^2)$. We write $\xi(f)$ for a pairing of $f \in C_0^r(\mathbb{R}^2)$ and $\xi \in (C_0^r)'(\mathbb{R}^2)$.

We usually use a variable $z = (t, x)$ as a point in $\mathbb{R}^2$ and write $dz = dtdx$. For $z = (t, x) \in \mathbb{R}^2$, we write $\|z\|_{\mathfrak{s}} = \sqrt{|t|} + |x|$. We also define $B_{\mathfrak{s}}(z, r) = \{\bar{z} \in \mathbb{R}^2 ; \|z - \bar{z}\|_{\mathfrak{s}} \le r\}$. For $z = (t, x) \in \mathbb{R}^2$, $\delta > 0$ and a function ρ on $\mathbb{R}^2$, we define ρ_z^δ by

$$\rho_z^\delta(\bar{z}) = \delta^{-3}\rho(\delta^{-2}(\bar{t} - t), \delta^{-1}(\bar{x} - x)) \quad (\bar{z} = (\bar{t}, \bar{x}) \in \mathbb{R}^2).$$

For a multiindex $k = (k_0, k_1) \in \mathbb{Z}_+^2$, we write $|k|_{\mathfrak{s}} = 2k_0 + k_1$ and $\partial^k = \partial_t^{k_0}\partial_x^{k_1}$. Especially, we write $' = \partial^{(0,1)}$. For $z = (t, x) \in \mathbb{R}^2$ and $k = (k_0, k_1) \in \mathbb{Z}_+^2$, we write $z^k = t^{k_0}x^{k_1}$.

Let $\alpha \in (0,1)$. We denote by $\mathcal{C}_{\mathfrak{s}}^{\alpha}(\mathbb{R}^2)$ the space of functions f on $\mathbb{R}^2$ such that for every compact set $\mathfrak{K} \subset \mathbb{R}^2$ we have

$$\|f\|_{\alpha;\mathfrak{K}} := \sup_{z,\bar{z}\in\mathfrak{K}} \frac{|f(z)-f(\bar{z})|}{\|z-\bar{z}\|_{\mathfrak{s}}^{\alpha}} < \infty.$$

We also denote by $\mathcal{C}_{\mathfrak{s}}^{\alpha}((0,\infty)\times\mathbb{R})$ the space of functions f on $(0,\infty)\times\mathbb{R}$ such that $\|f\|_{\alpha;\mathfrak{K}} < \infty$ for every compact set $\mathfrak{K} \subset (0,\infty)\times\mathbb{R}$.

Let $\alpha < 0$ and $r = \lceil -\alpha \rceil$ (r is the smallest integer such that $-\alpha < r$). We say that $\xi \in \mathcal{S}'(\mathbb{R}^2)$ belongs to $\mathcal{C}_{\mathfrak{s}}^{\alpha}(\mathbb{R}^2)$ if and only if it belongs to $(C_0^r)'(\mathbb{R}^2)$ and for every compact set $\mathfrak{K} \subset \mathbb{R}^2$ we have

$$\|\xi\|_{\alpha;\mathfrak{K}} := \sup_{z\in\mathfrak{K}} \sup_{\rho\in\mathcal{B}_0^r} \sup_{\delta\in(0,1]} \delta^{-\alpha}|\xi(\rho_z^\delta)| < \infty.$$

The operator $*_{t,x}$ denotes the convolution in (t,x). $*_t$ and $*_x$ denote the convolutions in t and x, respectively. The operator $*$ always denotes $*_{t,x}$.

For a function f on $\mathbb{R}^2$, we define $\overleftarrow{f}(t,x) = f(-t,-x)$.

For linear spaces A and B, we denote by $\mathcal{L}(A,B)$ the space of linear maps form A to B. If $A = B$, then we write $\mathcal{L}(A) = \mathcal{L}(A,A)$. $\mathrm{id} = \mathrm{id}_A \in \mathcal{L}(A)$ denotes the identity map on A. For a subset $S \subset A$, we denote by $\langle S \rangle$ the linear subspace of A spanned by S.

2.2. **Fractional derivatives of white noise.** For $\gamma > 0$, we write $H^\gamma(\mathbb{T}) := \{f \in L^2(\mathbb{T}) \,;\, \|f\|_{H^\gamma(\mathbb{T})} := (\sum_{n\in\mathbb{Z}}(2\pi|n|)^{2\gamma}|\mathcal{F}f(n)|^2)^{\frac{1}{2}} < \infty\}$. For $f,g \in H^\gamma(\mathbb{T})$, we define

$$(f,g)_{H^\gamma(\mathbb{T})} = \sum_{n\in\mathbb{Z}} (2\pi|n|)^{2\gamma}\mathcal{F}f(n)\overline{\mathcal{F}g(n)}.$$

We define the operator $(-\Delta)^{\frac{\gamma}{2}} : H^\gamma(\mathbb{T}) \to L^2(\mathbb{T})$ by

$$(-\Delta)^{\frac{\gamma}{2}} f(x) = \sum_{n\in\mathbb{Z}} (2\pi|n|)^{\gamma}\mathcal{F}f(n)e^{2\pi i n x} \quad (x \in \mathbb{R}).$$

For a function f on $\mathbb{R}\times\mathbb{T}$ such that $f(t,\cdot) \in H^\gamma(\mathbb{T})$ for every $t \in \mathbb{R}$, we write $\partial_x^\gamma f(t,x) = ((-\Delta)^{\frac{\gamma}{2}} f(t,\cdot))(x)$.

Let $\{W_h\}_{h\in L^2(\mathbb{R}\times\mathbb{T})}$ be a space-time white noise on $\mathbb{R}\times\mathbb{T}$ defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. That is, $\{W_h\}$ is a collection of centered Gaussian random variables such that $\mathbb{E}(W_h W_g) = (h,g)_{L^2(\mathbb{R}\times\mathbb{T})}$. For a function h on $\mathbb{R}\times\mathbb{T}$ such that $\int_{\mathbb{R}} \|h(t,\cdot)\|_{H^\gamma(\mathbb{T})}^2 dt < \infty$, we define $\partial_x^\gamma W_h := W_{\partial_x^\gamma h}$. Then we have

$$\mathbb{E}(\partial_x^\gamma W_h \partial_x^\gamma W_g) = \int_{\mathbb{R}} (h(t,\cdot), g(t,\cdot))_{H^\gamma(\mathbb{T})} dt.$$

We define the periodic extension of $\partial_x^\gamma W$.

Lemma 2.1. *Let $0 < \gamma < \frac{3}{2}$. Then for every $h \in \mathcal{S}(\mathbb{R}^2) \cup C_0^2(\mathbb{R}^2)$, we have $\int_{\mathbb{R}} \|\pi h(t,\cdot)\|_{H^\gamma(\mathbb{T})}^2 dt < \infty$.*

proof. Since $\mathcal{F}_{\mathbb{T}}(\pi h)(t,n) = \mathcal{F}_{\mathbb{R}} h(t,n)$ for every $n \in \mathbb{Z}$, we have

$$\begin{aligned}\int_{\mathbb{R}} \|\pi h(t,\cdot)\|_{H^\gamma(\mathbb{T})}^2 dt &= \int_{\mathbb{R}} \sum_n (2\pi|n|)^{2\gamma}|\mathcal{F}h(t,n)|^2 dt \\ &= \int_{\mathbb{R}} \sum_n (2\pi|n|)^{2\gamma}(4\pi^2n^2)^{-2}|\mathcal{F}(\partial_x^2 h)(t,n)|^2 dt\end{aligned}$$

$$\leq \sum_{n\neq 0}(2\pi|n|)^{2\gamma-4}\int_{\mathbb{R}}\|\partial_x^2 h(t,\cdot)\|_{L^1(\mathbb{R})}^2 dt.$$

The last term is finite because $\sum_{n\neq 0}|n|^{2\gamma-4}<\infty$. □

For $h\in\mathcal{S}(\mathbb{R}^2)\cup C_0^2(\mathbb{R}^2)$, we define $\partial_x^\gamma\xi(h):=\partial_x^\gamma W_{\pi h}$.

Lemma 2.2. *Let $0<\gamma<\frac{1}{2}$ and $-2<\alpha<-\frac{3}{2}-\gamma$. Then for every $p\geq 1$ and compact set $\mathfrak{K}\subset\mathbb{R}^2$, we have*

$$\mathbb{E}\|\partial_x^\gamma\xi\|_{\alpha;\mathfrak{K}}^p<\infty.$$

Furthermore, let $\rho\in C_0^\infty(\mathbb{R}^2)$ be a function such that $\int\rho(z)dz=1$, and for $\epsilon>0$ we define a function $\partial_x^\gamma\xi_\epsilon$ on $\mathbb{R}^2$ by $\partial_x^\gamma\xi_\epsilon(z):=\partial_x^\gamma\xi(\rho_z^\epsilon)$. Then for every $\kappa\in(0,-\frac{3}{2}-\gamma-\alpha)$, $p\geq 1$ and compact set $\mathfrak{K}\subset\mathbb{R}^2$, we have

$$\mathbb{E}\|\partial_x^\gamma\xi-\partial_x^\gamma\xi_\epsilon\|_{\alpha;\mathfrak{K}}^p\lesssim\epsilon^{\kappa p}.$$

proof. We consider a finite set Ψ of compactly supported and smooth functions $\psi\in C_0^2(\mathbb{R}^2)$ as in Section 3.1 of [3]. For $\psi\in\Psi$, $m\in\mathbb{Z}_+$ and $z=(t,x)\in\mathbb{R}^2$, we set

$$\psi_z^{m,\mathfrak{s}}(\bar{z})=2^{-\frac{3m}{2}}\psi_z^{2^{-m}}(\bar{z})=2^{\frac{3m}{2}}\psi(2^{2m}(\bar{t}-t),2^m(\bar{x}-x))\quad(\bar{z}=(\bar{t},\bar{x})).$$

From Proposition 3.20 of [3], for every $\alpha\in(-2,0)$ and compact set $\mathfrak{K}\subset\mathbb{R}^2$ we have

$$\|\partial_x^\gamma\xi\|_{\alpha;\mathfrak{K}}\lesssim_\Psi\sup_{\psi\in\Psi}\sup_{m\geq 0}\sup_{z\in\Lambda_m^{\mathfrak{s}}\cap\bar{\mathfrak{K}}}2^{\frac{3m}{2}+m\alpha}|\partial_x^\gamma\xi(\psi_z^{m,\mathfrak{s}})|.$$

Here $\Lambda_m^{\mathfrak{s}}:=\{(2^{-2m}k_0,2^{-m}k_1)\,;(k_0,k_1)\in\mathbb{Z}^2\}$, and $\bar{\mathfrak{K}}$ is the 1-fattening of $\mathfrak{K}$. Then for every $p\geq 1$ we have

$$\begin{aligned}\mathbb{E}\|\partial_x^\gamma\xi\|_{\alpha;\mathfrak{K}}^{2p}&\lesssim\sum_{\psi\in\Psi}\sum_{m\geq 0}\sum_{z\in\Lambda_m^{\mathfrak{s}}\cap\bar{\mathfrak{K}}}2^{(3+2\alpha)mp}\mathbb{E}|\partial_x^\gamma\xi(\psi_z^{m,\mathfrak{s}})|^{2p}\\&\lesssim\sum_{m\geq 0}2^{3m}2^{(3+2\alpha)mp}\mathbb{E}|\partial_x^\gamma\xi(\psi_z^{m,\mathfrak{s}})|^{2p}\\&\lesssim\sum_{m\geq 0}2^{3m}2^{(3+2\alpha)mp}(\mathbb{E}|\partial_x^\gamma\xi(\psi_z^{m,\mathfrak{s}})|^2)^p.\end{aligned}$$

Here we use the equivalence of moments for Gaussian random variables. By the definition of $\partial_x\xi$, we have

$$\begin{aligned}\mathbb{E}|\partial_x^\gamma\xi(\psi_{(t,x)}^{m,\mathfrak{s}})|^2&=\int_{\mathbb{R}}\|\pi\psi_{(t,x)}^{m,\mathfrak{s}}(\bar{t},\cdot)\|_{H^\gamma(\mathbb{T})}^2d\bar{t}=\int_{\mathbb{R}}\sum_n(2\pi|n|)^{2\gamma}|\mathcal{F}\psi_{(t,x)}^{m,\mathfrak{s}}(\bar{t},n)|^2d\bar{t}\\&=\int_{\mathbb{R}}\sum_n(2\pi|n|)^{2\gamma}\left|2^{\frac{m}{2}}\mathcal{F}\psi\left(2^{2m}(\bar{t}-t),\frac{n}{2^m}\right)e^{-2\pi inx}\right|^2d\bar{t}\\&\lesssim 2^{2m\gamma}\left\{\frac{1}{2^m}\int_{\mathbb{R}}\sum_n\left|\frac{n}{2^m}\right|^{2\gamma}\left|\mathcal{F}\psi\left(\bar{t},\frac{n}{2^m}\right)\right|^2d\bar{t}\right\}\\&\lesssim 2^{2m\gamma}\times\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\bar{x}|^{2\gamma}|\mathcal{F}\psi(\bar{t},\bar{x})|^2d\bar{x}\right)d\bar{t}\lesssim 2^{2m\gamma}.\end{aligned}$$

Hence for sufficiently large $p>1$ we have

$$\mathbb{E}||\partial_x^\gamma\xi||_{\alpha;\mathfrak{K}}^{2p}\lesssim\sum_{m\geq 0}2^{3m}2^{(3+2\alpha+2\gamma)mp}<\infty.$$

Next we prove the convergence result. Since $(\partial_x^\gamma \xi_\epsilon)(\psi_z^{m,\mathfrak{s}}) = \partial_x^\gamma \xi(\rho_\epsilon * \psi_z^{m,\mathfrak{s}})$ with $\rho_\epsilon = \rho_0^\epsilon$, we verify that

$$\int_{\mathbb{R}} \|\pi(\psi_z^{m,\mathfrak{s}} - \rho_\epsilon * \psi_z^{m,\mathfrak{s}})(t,\cdot)\|_{H^\gamma(\mathbb{T})}^2 dt \lesssim 2^{2m\gamma}(1 \wedge \epsilon^2 2^{2m}).$$

Since for every $h \in L^2(\mathbb{R}\times\mathbb{T})$ we have

$$\begin{aligned}
\int_{\mathbb{R}} \|\pi(\rho_\epsilon * h)(t,\cdot)\|_{H^\gamma(\mathbb{T})}^2 dt &= \int_{\mathbb{R}} \sum_n (2\pi|n|)^{2\gamma} |\mathcal{F}(\rho_\epsilon *_{t,x} h)(t,n)|^2 dt \\
&= \sum_n (2\pi|n|)^{2\gamma} \|\mathcal{F}\rho_\epsilon(\cdot,n) *_t \mathcal{F}h(\cdot,n)\|_{L^2(\mathbb{R})}^2 \\
&\le \sum_n (2\pi|n|)^{2\gamma} \|\mathcal{F}\rho_\epsilon(\cdot,n)\|_{L^1(\mathbb{R})}^2 \|\mathcal{F}h(\cdot,n)\|_{L^2(\mathbb{R})}^2 \\
&\le \|\rho\|_{L^1(\mathbb{R}^2)}^2 \int_{\mathbb{R}} \|\pi h(t,\cdot)\|_{H^\gamma(\mathbb{T})}^2 dt,
\end{aligned}$$

we have

$$\int_{\mathbb{R}} \|\pi(\psi_z^{m,\mathfrak{s}} - \rho_\epsilon * \psi_z^{m,\mathfrak{s}})(t,\cdot)\|_{H^\gamma(\mathbb{T})}^2 dt \lesssim \int_{\mathbb{R}} \|\pi\psi_z^{m,\mathfrak{s}}(t,\cdot)\|_{H^\gamma(\mathbb{T})}^2 dt \lesssim 2^{2m\gamma}.$$

Assume that $\epsilon 2^m \le 1$. We decompose the integration as follows.

$$\begin{aligned}
&\int_{\mathbb{R}} \|\pi(\psi_{(t,x)}^{m,\mathfrak{s}} - \rho_\epsilon * \psi_{(t,x)}^{m,\mathfrak{s}})(t,\cdot)\|_{H^\gamma(\mathbb{T}))}^2 \\
&= \int_{\mathbb{R}} \sum_n (2\pi|n|)^{2\gamma} |\mathcal{F}(\psi_{(t,x)}^{m,\mathfrak{s}} - \rho_\epsilon * \psi_{(t,x)}^{m,\mathfrak{s}})(\bar{t},n)|^2 d\bar{t} \\
&\lesssim \int_{\mathbb{R}} \sum_n (2\pi|n|)^{2\gamma} \left| \mathcal{F}\psi_{(t,x)}^{m,\mathfrak{s}}(\bar{t},n) \left(1 - \int_{\mathbb{R}} \mathcal{F}\rho_\epsilon(\bar{t}-\bar{s},n) d\bar{s}\right)\right|^2 d\bar{t} \\
&\qquad + \int_{\mathbb{R}} \sum_n (2\pi|n|)^{2\gamma} \left| \int_{\mathbb{R}} (\mathcal{F}\psi_{(t,x)}^{m,\mathfrak{s}}(\bar{t},n) - \mathcal{F}\psi_{(t,x)}^{m,\mathfrak{s}}(\bar{s},n)) \mathcal{F}\rho_\epsilon(\bar{t}-\bar{s},n) d\bar{s}\right|^2 d\bar{t} \\
&=: I_1 + I_2.
\end{aligned}$$

For I_1, since

$$\begin{aligned}
\left|1 - \int_{\mathbb{R}} \mathcal{F}\rho_\epsilon(t,n) dt\right| = \left|1 - \int_{\mathbb{R}} \mathcal{F}\rho(t,\epsilon n) dt\right| &= \left|\int_{\mathbb{R}^2} \rho(t,x)(1 - e^{-2\pi i\epsilon n x}) dt dx\right| \\
&\lesssim \epsilon|n| \int_{\mathbb{R}^2} |x|\ |\rho(t,x)| dt dx \lesssim \epsilon|n|,
\end{aligned}$$

we have

$$\begin{aligned}
I_1 &\lesssim \int_{\mathbb{R}} \sum_n (2\pi|n|)^{2\gamma} \epsilon^2 |n|^2 \left| 2^{\frac{m}{2}} \mathcal{F}\psi\left(2^{2m}(\bar{t}-t), \frac{n}{2^m}\right) e^{-2\pi i n x}\right|^2 d\bar{t} \\
&\lesssim \epsilon^2 2^{2m} 2^{2m\gamma} \left(\frac{1}{2^m} \int_{\mathbb{R}} \sum_n \left|\frac{n}{2^m}\right|^{2\gamma+2} \left|\mathcal{F}\psi\left(\bar{t}, \frac{n}{2^m}\right)\right|^2 d\bar{t}\right) \lesssim \epsilon^2 2^{2m} 2^{2m\gamma}.
\end{aligned}$$

For I_2, since $\epsilon \le 2^{-m}$, there exists $C > 0$ such that the integrand is supported in $|\bar{t}-t| \le C2^{-2m}$. Hence we have

$$I_2 = 2^m \int_{|\bar{t}-t|\le C2^{-2m}} \sum_n (2\pi|n|)^{2\gamma} \bigg| \int_{\mathbb{R}} \{\mathcal{F}\psi(2^{2m}(\bar{t}-t), \frac{n}{2^m}) - \mathcal{F}\psi(2^{2m}(\bar{s}-t, \frac{n}{2^m}))\}$$

$$\times\, \epsilon^{-2}\mathcal{F}\rho(\epsilon^{-2}(\bar t-\bar s,\epsilon n))d\bar s\Big|^2 d\bar t$$

$$\begin{aligned}
&\lesssim 2^m\sum_n (2\pi|n|)^{2\gamma}\Big\|\partial_t\mathcal{F}\psi(\cdot,\frac{n}{2^m})\Big\|^2_{L^\infty(\mathbb{R})}\\
&\qquad\qquad\times\int_{|\bar t-t|\le C2^{-2m}}\Big|\int \epsilon^{-2}2^m|\bar t-\bar s|\mathcal{F}\rho(\epsilon^{-2}(\bar t-\bar s),\epsilon n)d\bar s\Big|^2 d\bar t\\
&\lesssim \epsilon^4 2^{4m}2^{2m\gamma}\frac{1}{2^m}\sum_n\Big|\frac{n}{2^m}\Big|^{2\gamma}\Big\|\partial_t\mathcal{F}\psi(\cdot,\frac{n}{2^m})\Big\|^2_{L^\infty(\mathbb{R})}\lesssim \epsilon^4 2^{4m}2^{2m\gamma}\le \epsilon^2 2^{2m}2^{2m\gamma}.
\end{aligned}$$

Therefore for sufficiently large $p>1$ we have

$$\begin{aligned}
\mathbb{E}\|\partial_x^\gamma\xi-\partial_x^\gamma\xi_\epsilon\|^{2p}_{\alpha;\mathfrak{K}} &\lesssim \sum_{m\ge 0}2^{3m}2^{(3+2\alpha+2\gamma)mp}(1\wedge\epsilon^2 2^{2m})^p\\
&\lesssim \epsilon^{-3-(3+2\alpha+2\gamma)p}=\epsilon^{2p(-\frac32-\alpha-\gamma-\frac{3}{2p})}.
\end{aligned}$$

□

2.3. **Fractional derivatives on** $\mathbb{R}$**.** For $\gamma>0$, we write $H^\gamma(\mathbb{R}):=\{f\in L^2(\mathbb{R})\,;\int_{\mathbb{R}}|\xi|^{2\gamma}|\mathcal{F}f(\xi)|^2<\infty\}$ and define the map $(-\Delta)^{\frac{\gamma}{2}}:H^\gamma(\mathbb{R})\to L^2(\mathbb{R})$ by

$$(-\Delta)^{\frac{\gamma}{2}}f(x)=\int_{\mathbb{R}}(2\pi|\xi|)^\gamma\mathcal{F}f(\xi)e^{2\pi i\xi x}d\xi\quad(x\in\mathbb{R}).$$

For a function f on $\mathbb{R}^2$ such that $f(t,\cdot)\in H^\gamma(\mathbb{R})$ for every $t\in\mathbb{R}$, we write $\partial_x^\gamma f(t,x)=((-\Delta)^{\frac{\gamma}{2}}f(t,\cdot))(x)$.

$\mathcal{S}(\mathbb{R})$ denotes the space of Schwartz functions on $\mathbb{R}$.

Lemma 2.3 (Proposition 3.3 of [1])**.** *Let $0<\gamma<1$. Then there exists a constant C_γ such that for every $f\in\mathcal{S}(\mathbb{R})$ we have*

$$(-\Delta)^{\frac{\gamma}{2}}f(x)=C_\gamma\int_{\mathbb{R}}\frac{f(x)-f(y)}{|x-y|^{1+\gamma}}dy.$$

Furthermore, we have the estimate $\|(-\Delta)^{\frac{\gamma}{2}}f\|_{L^\infty}\lesssim\|f\|_{L^\infty}+\|f'\|_{L^\infty}$.

Lemma 2.4. *Let $0<\gamma<1$. For every $f\in\mathcal{S}(\mathbb{R})$, there exists a constant C_f such that we have the inequality*

$$|(-\Delta)^{\frac{\gamma}{2}}f(x)|\le C_f|x|^{-1-\gamma},$$

where we have

$$C_f\sim\|\,|\cdot|^{1+\gamma}f\,\|_{L^\infty}+\|\,|\cdot|^{1+\gamma}f'\,\|_{L^\infty}+\|f\|_{L^1}+\|\,|\cdot|^{1+\gamma}f\,\|_{L^1}.$$

proof. From Lemma 2.3, we have

$$\begin{aligned}
|(-\Delta)^{\frac{\gamma}{2}}f(x)| &\le C_\gamma\Bigg(\Big|\int_{|x-y|\le1}\frac{f(x)-f(y)}{|x-y|^{1+\gamma}}dy\Big|+\Big|\int_{|x-y|>1}\frac{f(x)}{|x-y|^{1+\gamma}}dy\Big|\\
&\qquad\qquad+\Big|\int_{|x-y|>1}\frac{f(y)}{|x-y|^{1+\gamma}}dy\Big|\Bigg)\\
&=: I_1+I_2+I_3.
\end{aligned}\tag{2.1}$$

For the first and second term, for every $n > 0$ we have

$$I_1 \lesssim \int_{|x-y|\leq 1} \frac{1}{|x-y|^\gamma} \sup_{|x-\bar{y}|\leq 1} |f'(\bar{y})| dy \lesssim \sup_{|x-\bar{y}|\leq 1} |f'(\bar{y})| \lesssim |x|^{-n},$$
$$I_2 \lesssim \int_{|x-y|>1} \frac{1}{|x-y|^{1+\gamma}} dy \; |f(x)| \lesssim |f(x)| \lesssim |x|^{-n}.$$

For the third term, we have

$$\begin{aligned} I_3 &\lesssim \int_{|x-y|>1} \frac{|f(y)|}{|x-y|^{1+\gamma}} dy \\ &\lesssim \int_{|x-y|>1} \frac{|x-y|^{1+\gamma}|f(y)|}{|x-y|^{1+\gamma}} dy + \int_{|x-y|>1} \frac{|y|^{1+\gamma}|f(y)|}{|x-y|^{1+\gamma}} dy \\ &\lesssim \int_{\mathbb{R}} |f(y)| dy + \int_{\mathbb{R}} |y|^{1+\gamma}|f(y)| dy < \infty. \end{aligned}$$

□

Lemma 2.5. *Let $0 < \gamma < 1$. For every $f \in \mathcal{S}(\mathbb{R})$ and $k \in \mathbb{Z}_+$, there exists a constant $C_{f,k}$ such that we have*

$$|(-\Delta)^{\frac{\gamma}{2}} f^{(k)}(x)| \leq C_{f,k} |x|^{-1-k-\gamma},$$

where we have

$$C_{f,k} \sim \sum_{l=0}^{k+1} \| \, |\cdot|^{1+k+\gamma} f^{(l)} \, \|_{L^\infty} + \|f\|_{L^1} + \| \, |\cdot|^{1+k+\gamma} f \, \|_{L^1}.$$

proof. The proof is similar to Lemma 2.4. In the decomposition (2.1), I_1 and I_2 decrease sufficiently fast. It remains to show that

$$\left| \int_{|x-y|>1} \frac{f^{(k)}(y)}{|x-y|^{1+\gamma}} dy \right| \lesssim |x|^{-1-k-\gamma}.$$

Using integration by parts, we obtain

$$\begin{aligned} \int_{|x-y|>1} \frac{f^{(k)}(y)}{|x-y|^{1+\gamma}} dy = &- \sum_{l=1}^{k} \left(\prod_{m=1}^{l-1} (m+\gamma) \right) (f^{(k-l)}(x+1) + (-1)^l f^{(k-l)}(x-1)) \\ &+ \left(\prod_{m=1}^{k} (m+\gamma) \right) \int_{|x-y|>1} (\mathrm{sign}|x-y|)^k \frac{f(y)}{|x-y|^{1+k+\gamma}} dy. \end{aligned}$$

The first term decreases sufficiently fast. We have the second term $\lesssim |x|^{-1-k-\gamma}$, similarly to before. □

3. Regularity structures

We recall some concepts in the theory of regularity structures from [3].

3.1. **Definitions.**

Definition 3.1. *We say that a triplet* $\mathcal{T} = (A, T, G)$ *is a* regularity structure *with* index set A, model space T *and* structure group G, *if and only if*

(1) A *is a countable subset of* $\mathbb{R}$, $0 \in A$, *bounded from below, and has no accumulation points.*
(2) $T = \bigoplus_{\alpha \in A} T_\alpha$ *is a direct sum of normed spaces. Furthermore,* $\dim T_0 = 1$ *and its unit vector is denoted by* $\mathbf{1}$.
(3) G *is a subgroup of* $\mathcal{L}(T)$ *such that, for every* $\Gamma \in G$, $\alpha \in A$, *and* $\tau \in T_\alpha$ *we have*
$$\Gamma\tau - \tau \in \bigoplus_{\beta < \alpha} T_\beta.$$
Furthermore, $\Gamma\mathbf{1} = \mathbf{1}$ *for every* $\Gamma \in G$.

The norm of T_α is denoted by $\|\cdot\|_\alpha$. For an element $\tau \in T$, we write $\|\tau\|_\alpha = \|\tau_\alpha\|_\alpha$, where τ_α is the component of τ in T_α. For $\beta > 0$, we write $T_\beta^- = \bigoplus_{\alpha < \beta} T_\alpha$.

Definition 3.2. *Let* $\mathcal{T} = (A, T, G)$ *be a regularity structure. Let* B *be a subset of* A *such that* $0 \in B$, *and* $V_\beta \neq \{0\}$ *be a linear subspace of* T_β *for each* $\beta \in B$. *We say that* $V = \bigoplus_{\beta \in B} V_\beta$ *is a* sector *of* $\mathcal{T}$, *if and only if* $\Gamma V \subset V$ *for every* $\Gamma \in G$.

We say that $\beta = \min B$ is the *regularity* of V. One important example of a regularity structure is the structure generated by polynomials. Let X_0, X_1 be dummy variables and $T = \mathbb{R}[X_0, X_1]$ be the linear space of polynomials in X_0, X_1. For a multiindex $k = (k_0, k_1) \in \mathbb{Z}_+^2$, we write $X^k = X_0^{k_0} X_1^{k_1}$. We denote by $\mathbf{1} = X^{(0,0)}$.

Definition 3.3. *The polynomial regularity structure* (A, T, G) *consists of the following elements.*

(1) $A = \mathbb{Z}_+$.
(2) $T = \bigoplus_{n \in \mathbb{Z}_+} T_n$, *where* $T_n = \langle X^k \,;\, |k|_{\mathfrak{s}} = n \rangle$.
(3) $G = \{\Gamma_h \,;\, h \in \mathbb{R}^2\}$, *where* Γ_h *is defined by* $\Gamma_h X^k = (X + h\mathbf{1})^k$.

We define models for a regularity structure.

Definition 3.4. *Let* $\mathcal{T} = (A, T, G)$ *be a regularity structure with regularity* $\alpha_0 \leq 0$, *and* $r = \lceil -\alpha_0 \rceil$ *We say that* $Z = (\Pi, \Gamma)$ *is a* model *for* $\mathcal{T}$, *if and only if*

(1) Γ *is a map* $\mathbb{R}^2 \times \mathbb{R}^2 \to G$ *such that* $\Gamma_{z,z} = \mathrm{id}_T$ *and* $\Gamma_{z,\bar{z}}\Gamma_{\bar{z},\bar{\bar{z}}} = \Gamma_{z,\bar{\bar{z}}}$ *for every* $z, \bar{z}, \bar{\bar{z}} \in \mathbb{R}^2$. *Furthermore, for every* $\gamma > 0$ *and compact set* $\mathfrak{K} \subset \mathbb{R}^2$, *we have*
$$\|\Gamma\|_{\gamma;\mathfrak{K}} := \sup\left\{ \frac{\|\Gamma_{z,\bar{z}}\tau\|_\beta}{\|\tau\|_\alpha \|z - \bar{z}\|_{\mathfrak{s}}^{\alpha - \beta}} \,;\, \beta < \alpha < \gamma,\, \tau \in T_\alpha,\, (z, \bar{z}) \in \mathfrak{K}^2 \right\} < \infty.$$
(2) Π *is a map* $\mathbb{R}^2 \to \mathcal{L}(\mathcal{T}, \mathcal{S}'(\mathbb{R}^2))$ *such that* $\Pi_z \Gamma_{z,\bar{z}} = \Pi_{\bar{z}}$ *for every* $z, \bar{z} \in \mathbb{R}^2$. *Furthermore, for every* $\gamma > 0$ *and compact set* $\mathfrak{K} \subset \mathbb{R}^2$, *we have*
$$\|\Pi\|_{\gamma;\mathfrak{K}} := \sup\left\{ \frac{|(\Pi_z\tau)(\rho_z^\delta)|}{\|\tau\|_\alpha \delta^\alpha} \,;\, \alpha < \gamma,\, \tau \in T_\alpha,\, z \in \mathfrak{K},\, \rho \in \mathcal{B}_0^r,\, \delta \in (0, 1] \right\} < \infty.$$

For models $Z = (\Pi, \Gamma)$ and $\bar{Z} = (\bar{\Pi}, \bar{\Gamma})$ on $\mathcal{T}$, we write
$$\|Z\|_{\gamma;\mathfrak{K}} = \|\Gamma\|_{\gamma;\mathfrak{K}} + \|\Pi\|_{\gamma;\mathfrak{K}}, \quad \|Z - \bar{Z}\|_{\gamma;\mathfrak{K}} = \|\Gamma - \bar{\Gamma}\|_{\gamma;\mathfrak{K}} + \|\Pi - \bar{\Pi}\|_{\gamma;\mathfrak{K}}.$$

3.2. **Regularity structures for (1.1).** We construct regularity structures for (1.1), following Section 8.1 of [3]. We use dummy variables Ξ (noise), $\mathbf{1}$ (constant), X_1 (time variable), X_2 (spatial variable), and abstract operators $\{\mathcal{I}_k\}_{k\in\mathbb{Z}_+^2}$ (convolution with k-th derivative of heat kernel). Especially we write $\mathcal{I}=\mathcal{I}_{(0,0)}$ and $\mathcal{I}'=\mathcal{I}_{(0,1)}$. We define $\widetilde{\mathcal{F}}$ as the minimal set of variables such that

$$(1)\ \Xi,\mathbf{1},X_0,X_1\in\widetilde{\mathcal{F}},\quad (2)\ \tau,\bar\tau\in\widetilde{\mathcal{F}}\Rightarrow\tau\bar\tau\in\widetilde{\mathcal{F}},$$
$$(3)\ \tau\in\widetilde{\mathcal{F}}\setminus\{X^l\,;l\in\mathbb{Z}_+^2\},k\in\mathbb{Z}_+^2\Rightarrow\mathcal{I}_k\tau\in\widetilde{\mathcal{F}}.$$

In (2), we postulate that $\tau\bar\tau=\bar\tau\tau$. For a fixed number $\alpha_0\in\mathbb{R}$, we can define the homogeneity (Besov index in parabolic scaling) of each variable by

$$|\Xi|_{\mathfrak{s}}=\alpha_0,\quad |\mathbf{1}|_{\mathfrak{s}}=0,\quad |X_0|_{\mathfrak{s}}=2,\quad |X_1|_{\mathfrak{s}}=1$$
$$|\tau\bar\tau|_{\mathfrak{s}}=|\tau|_{\mathfrak{s}}+|\bar\tau|_{\mathfrak{s}}\quad |\mathcal{I}_k\tau|_{\mathfrak{s}}=|\tau|_{\mathfrak{s}}+2-|k|_{\mathfrak{s}}.$$

We define the abstract operator ∂ acting on $\{\mathcal{I}\tau\}\cup\{X^k\}$ by

$$\partial\mathcal{I}\tau=\mathcal{I}'\tau,\quad \partial(X_0^{k_0}X_1^{k_1})=\mathbf{1}_{k_1\neq0}k_1X_0^{k_0}X_1^{k_1-1}.$$

We define the sets $\mathcal{U}_n$ and $\mathcal{V}_n$ for $n\in\mathbb{Z}_+$ recursively by

$$\mathcal{U}_0=\mathcal{V}_0=\{X^k\,;k\in\mathbb{Z}_+^2\},$$
$$\mathcal{V}_n=\{\Xi\}\cup\{\partial\tau_1\partial\tau_2\,;\tau_1,\tau_2\in\mathcal{U}_{n-1}\},\ \ \mathcal{U}_n=\mathcal{U}_{n-1}\cup\{\mathcal{I}\tau\,;\tau\in\mathcal{V}_n\}\quad (n\geq1).$$

We set $\mathcal{U}=\bigcup_{n\geq0}\mathcal{U}_n$, $\mathcal{V}=\bigcup_{n\geq0}\mathcal{V}_n$ and $\mathcal{F}=\mathcal{U}\cup\mathcal{V}$. We define

$$T_\alpha=\langle\tau\in\mathcal{F}\,;|\tau|_{\mathfrak{s}}=\alpha\rangle,\quad T=\langle\mathcal{F}\rangle,\quad U=\langle\mathcal{U}\rangle,\quad V=\langle\mathcal{V}\rangle.$$

In order to define T as a model space of a regularity structure, the set $\{|\tau|_{\mathfrak{s}}\,;\tau\in\mathcal{F}\}$ must be bounded from below. A nonliner SPDE is called *subcritical*, if nonlinear terms formally disappear in some scaling which keeps the linear part and the noise term invariant. This is equivalent to the property that all variables except Ξ defined as above have homogeneities strictly greater than $|\Xi|_{\mathfrak{s}}$ (Assumption 8.3 of [3]). In the present case, this is equivalent to $|\mathcal{I}'(\Xi)|_{\mathfrak{s}}=2(1+\alpha_0)>\alpha_0\Leftrightarrow\alpha_0>-2$. Additionally, we should assume $-2<\alpha_0<-\frac{3}{2}-\gamma$ from Lemma 2.2, which implies $0\leq\gamma<\frac{1}{2}$.

Lemma 3.1 (Lemma 8.10 of [3]). *Let $0\leq\gamma<\frac12$ and $-2<\alpha_0<-\frac32-\gamma$. Then the set $\{\tau\in\mathcal{F}\,;|\tau|_{\mathfrak{s}}<r\}$ is finite for every $r>0$.*

3.3. **Structure group.** We define the structure group on T following Section 8.1 of [3]. Instead of $\{\mathcal{I}_k\}_{k\in\mathbb{Z}_+^2}$, we use operators $\{\mathcal{J}_k\}_{k\in\mathbb{Z}_+^2}$. We write $\mathcal{J}=\mathcal{J}_{(0,0)}$ and $\mathcal{J}'=\mathcal{J}_{(0,1)}$. We define $\mathcal{F}^+$ as the minimal set of variables such that

$$(1)\ \mathbf{1},X_0,X_1\in\mathcal{F}^+,\quad (2)\ \tau,\bar\tau\in\mathcal{F}^+\Rightarrow\tau\bar\tau\in\mathcal{F}^+,$$
$$(3)\ \tau\in\mathcal{F}\setminus\{X^l\,;l\in\mathbb{Z}_+^2\},k\in\mathbb{Z}_+^2,|\tau|_{\mathfrak{s}}+2-|k|_{\mathfrak{s}}>0\Rightarrow\mathcal{J}_k\tau\in\mathcal{F}^+.$$

We can define the homogeneity of each variable by

$$|\mathbf{1}|_{\mathfrak{s}}=0,\quad |X_0|_{\mathfrak{s}}=2,\quad |X_1|_{\mathfrak{s}}=1,\quad |\tau\bar\tau|_{\mathfrak{s}}=|\tau|_{\mathfrak{s}}+|\bar\tau|_{\mathfrak{s}},\quad |\mathcal{J}_k\tau|_{\mathfrak{s}}=|\tau|_{\mathfrak{s}}+2-|k|_{\mathfrak{s}}.$$

We write $\mathcal{H}=\langle\mathcal{F}\rangle$ and $\mathcal{H}^+=\langle\mathcal{F}^+\rangle$. We have a natural linear map $\hat{\mathcal{J}}_k:\mathcal{H}\to\mathcal{H}^+$ defined by the linear extension of

$$\hat{\mathcal{J}}_k(\tau)=\begin{cases}\mathcal{J}_k\tau & |\tau|_{\mathfrak{s}}+2-|k|_{\mathfrak{s}}>0\\ 0 & |\tau|_{\mathfrak{s}}+2-|k|_{\mathfrak{s}}\leq0\end{cases},\quad \tau\in\mathcal{F}.$$

We simply write again $\mathcal{J}_k$ instead of $\hat{\mathcal{J}}_k$.

We define the linear map $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}^+$ by

$$\Delta \mathbf{1} = \mathbf{1} \otimes \mathbf{1}, \quad \Delta X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i \ (i = 0, 1), \quad \Delta \Xi = \Xi \otimes \mathbf{1},$$

$$\Delta(\tau\bar{\tau}) = (\Delta\tau)(\Delta\bar{\tau}), \quad \Delta(\mathcal{I}_k\tau) = (\mathcal{I}_k \otimes \mathrm{id})\Delta\tau + \sum_{l,m} \frac{X^l}{l!} \otimes \frac{X^m}{m!}\mathcal{J}_{k+l+m}\tau.$$

We also define the linear map $\Delta^+ : \mathcal{H}^+ \to \mathcal{H}^+ \otimes \mathcal{H}^+$ by

$$\Delta^+ \mathbf{1} = \mathbf{1} \otimes \mathbf{1}, \quad \Delta^+ X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i \ (i = 0, 1),$$

$$\Delta^+(\tau\bar{\tau}) = (\Delta^+\tau)(\Delta^+\bar{\tau}), \quad \Delta^+(\mathcal{J}_k\tau) = \sum_l \left(\mathcal{J}_{k+l} \otimes \frac{(-X)^l}{l!} \right) \Delta\tau + \mathbf{1} \otimes \mathcal{J}_k\tau.$$

Furthermore, we define the linear map $\mathcal{A} : \mathcal{H}^+ \to \mathcal{H}^+$ by

$$\mathcal{A}\mathbf{1} = \mathbf{1}, \quad \mathcal{A}X_i = -X_i \ (i = 0, 1), \quad \mathcal{A}(\tau\bar{\tau}) = (\mathcal{A}\tau)(\mathcal{A}\bar{\tau}),$$

$$\mathcal{A}\mathcal{J}_k\tau = -\sum_l \mathcal{M}\left(\mathcal{J}_{k+l} \otimes \frac{X^l}{l!}\mathcal{A} \right) \Delta\tau.$$

Here $\mathcal{M} : \mathcal{H}^+ \times \mathcal{H}^+ \to \mathcal{H}^+$ is the multiplication operator defined by $\mathcal{M}(\tau \otimes \bar{\tau}) = \tau\bar{\tau}$. We define the product $\circ$ on $(\mathcal{H}^+)^*$ by

$$(g \circ \bar{g})(\tau) = (g \otimes \bar{g})(\Delta^+\tau) \quad (g, \bar{g} \in (\mathcal{H}^+)^*, \tau \in \mathcal{H}^+).$$

Lemma 3.2 (Theorem 8.16 of [3]). *$\mathcal{H}^+$ is a Hopf algebra with the antipode $\mathcal{A}$. $\mathcal{H}$ is a comodule over $\mathcal{H}^+$.*

We denote by G the set of algebra homomorphisms $g : \mathcal{H}^+ \to \mathbb{R}$. Then G is a group with the product $\circ$. The inverse of $g \in G$ is given by $g^{-1} = g\mathcal{A}$. For $g \in G$, we define the operator $\Gamma_g \in \mathcal{L}(T)$ by

$$\Gamma_g\tau = (\mathrm{id} \otimes g)\Delta\tau \quad (\tau \in T),$$

where we identify $T \otimes \mathbb{R}$ with T by $\tau \otimes a \mapsto a\tau$. Since $g \mapsto \Gamma_g$ is a group homomorphism (Proposition 8.19 of [3]), we can identify G with $\{\Gamma_g\, ; g \in G\}$.

Lemma 3.3 (Theorem 8.24 of [3]). *Let $0 \le \gamma < \frac{1}{2}$, $-2 < \alpha_0 < -\frac{3}{2} - \gamma$ and $A = \{|\tau|_\mathfrak{s}\, ; \tau \in \mathcal{F}\}$. Then (A, T, G) is a regularity structure.*

Given $r > 0$, obviously

$$\mathcal{T}^{(r)} = (A \cap (-\infty, r), T_r^-, G|_{T_r^-})$$

is also a regularity structure.

3.4. **Admissible models.** Fix $C, r > 0$. Let $K : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ be a function such that

(1) K is smooth except at 0, and supported in $B_\mathfrak{s}(0, C)$.
(2) $K(t, \cdot) = 0$ for $t \le 0$, and $K(t, x) = K(t, -x)$ for every $(t, x) \in \mathbb{R}^2$.
(3) For every $(t, x) \in B_\mathfrak{s}(0, \frac{C}{2})$, we have

$$K(t, x) = \mathbf{1}_{t>0} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

(4) For every multiindex $k=(k_1,k_2)\in\mathbb{Z}_+^2$ with $|k|_{\mathfrak{s}}\le r$, we have

$$\iint K(t,x)t^{k_1}x^{k_2}dtdx=0.$$

See Lemma 5.5 of [3]. It is sufficient to take small C, so that all variables in graphical notations in Sections 5-8 are in $B_{\mathfrak{s}}(0,\frac{1}{4})$.

Definition 3.5. *We say that a model (Π,Γ) on $\mathcal{T}^{(r)}$ is* admissible*, if and only if for every $z=(t,x),\bar{z}=(\bar{t},\bar{x})\in\mathbb{R}^2$, multiindex $k\in\mathbb{Z}_+^2$ and $\tau\in\mathcal{F}$, Π satisfies*

$$(\Pi_z\mathcal{I}_k\tau)(\bar{z})=\partial^kK*(\Pi_z\tau)(\bar{z})+\sum_l\frac{(\bar{z}-z)^l}{l!}f_z(\mathcal{J}_{k+l}\tau),\quad(\Pi_zX^k)(\bar{z})=(\bar{z}-z)^k,$$

and Γ satisfies $\Gamma_{z\bar{z}}=(\Gamma_{f_z})^{-1}\Gamma_{f_{\bar{z}}}$. Here $\{f_z\in G\,;z\in\mathbb{R}^2\}$ is a family defined by

$$f_z(X_0)=-t,\ f_z(X_1)=-x,$$
$$f_z(\mathcal{J}_k\tau)=-\partial^kK*(\Pi_z\tau)(z)\quad(|\tau|_{\mathfrak{s}}+2-|k|_{\mathfrak{s}}>0).$$

From the definition of admissible models, the map $\Pi_z(\Gamma_{f_z})^{-1}:\mathcal{H}\to\mathcal{S}'(\mathbb{R}^2)$ is independent to z. Hence we can write $\mathbf{\Pi}=\Pi_z(\Gamma_{f_z})^{-1}$. If (Π,Γ) is admissible, for every $z=(t,x)\in\mathbb{R}^2$, $k\in\mathbb{Z}_+^2$ and $\tau\in\mathcal{F}$ we have

$$(\mathbf{\Pi}\mathbf{1})(z)=1,\quad(\mathbf{\Pi}X_0)(z)=t,\quad(\mathbf{\Pi}X_1)(z)=x,\quad(\mathbf{\Pi}\mathcal{I}_k\tau)(z)=\partial^kK*(\mathbf{\Pi}\tau)(z).$$

Conversely, if a linear map $\mathbf{\Pi}:\mathcal{H}\to\mathcal{S}'(\mathbb{R}^2)$ satisfies these conditions, and a family $\{f_z\,;z\in G\}$ satisfies

$$f_z(X_0)=-t,\quad f_z(X_1)=-x,$$
$$f_z(\mathcal{J}_k\tau)=-\partial^kK*((\mathbf{\Pi}\otimes f_z)\Delta\tau)(z)\quad(|\tau|_{\mathfrak{s}}+2-|k|_{\mathfrak{s}}>0),$$

then an admissible model (Π,Γ) is uniquely determined by

$$\Pi_z=(\mathbf{\Pi}\otimes f_z)\Delta,\quad\Gamma_{z\bar{z}}=(\Gamma_{f_z})^{-1}\Gamma_{f_{\bar{z}}}.$$

Definition 3.6. *We say that a model (Π,Γ) on $\mathcal{T}^{(r)}$ is* periodic (in x)*, if and only if*

$$(\Pi_{(t,x+n)}\tau)(\varphi_n)=(\Pi_{(t,x)}\tau)(\varphi),\quad\Gamma_{(t,x+n),(\bar{t},\bar{x}+n)}=\Gamma_{(t,x),(\bar{t},\bar{x})},$$

for every $(t,x),(\bar{t},\bar{x})\in\mathbb{R}^2$, $n,\bar{n}\in\mathbb{Z}$ and $\varphi\in\mathcal{S}(\mathbb{R}^2)$. Here $\varphi_n(t,x)=\varphi(t,x-n)$.

3.5. **Modelled distributions.** Following Section 6 of [3], we define modelled distributions with singularity at $t=0$. We write $P=\{(t,x)\in\mathbb{R}^2\,;t=0\}$ and

$$\|(t,x)\|_P=1\wedge\sqrt{|t|},\quad\|z,\bar{z}\|_P=\|z\|_P\wedge\|\bar{z}\|_P.$$

For a subset $\mathfrak{K}\subset\mathbb{R}^2$, we write

$$\mathfrak{K}_P:=\{(z,\bar{z})\in(\mathfrak{K}\setminus P)^2\,;z\neq\bar{z},\|z-\bar{z}\|_{\mathfrak{s}}\le\|z,\bar{z}\|_P\}.$$

Definition 3.7. *Let $Z=(\Pi,\Gamma)$ be a model on $\mathcal{T}^{(r)}$, $\theta>0$ and $\eta\in\mathbb{R}$. For a T_β^--valued function f on $\mathbb{R}^2$, and a compact set $\mathfrak{K}\subset\mathbb{R}^2$, we define*

$$\|f\|_{\theta,\eta;\mathfrak{K}}=\sup_{z\in\mathfrak{K}\setminus P}\sup_{l<\theta}\frac{\|f(z)\|_l}{\|z\|_P^{(\eta-l)\wedge0}},$$
$$|\!|\!|f|\!|\!|_{\theta,\eta;\mathfrak{K}}=\|f\|_{\theta,\eta;\mathfrak{K}}+\sup_{(z,\bar{z})\in\mathfrak{K}_P}\sup_{l<\theta}\frac{\|f(z)-\Gamma_{z\bar{z}}f(\bar{z})\|_l}{\|z-\bar{z}\|_{\mathfrak{s}}^{\theta-l}\|z,\bar{z}\|_P^{\eta-\theta}}.$$

We write $f \in \mathcal{D}_P^{\theta,\eta} = \mathcal{D}_P^{\theta,\eta}(Z)$ if and only if $\|f\|_{\theta,\eta;\mathfrak{K}} < \infty$ for every compact subset $\mathfrak{K} \subset \mathbb{R}^2$.

If f takes value in a sector W of $\mathcal{T}^{(r)}$, we write $f \in \mathcal{D}_P^{\theta,\eta}(W;Z)$. For models Z, $\bar{Z}$ and $f \in \mathcal{D}_P^{\theta,\eta}(W;Z)$, $\bar{f} \in \mathcal{D}_P^{\theta,\eta}(W;\bar{Z})$, we define

$$\|f;\bar{f}\|_{\theta,\eta;\mathfrak{K}} = \|f-\bar{f}\|_{\theta,\eta;\mathfrak{K}} + \sup_{(z,\bar{z})\in\mathfrak{K}_P} \sup_{l<\theta} \frac{\|f(z)-\bar{f}(z)-\Gamma_{z\bar{z}}f(\bar{z})+\bar{\Gamma}_{z\bar{z}}\bar{f}(\bar{z})\|_l}{\|z-\bar{z}\|_{\mathfrak{s}}^{\theta-l}\|z,\bar{z}\|_P^{\eta-\theta}}.$$

We denote by $\mathcal{M} \ltimes \mathcal{D}_P^{\theta,\eta}$ the set of all pairs (Z,f) of a model Z and $f \in \mathcal{D}_P^{\theta,\eta}(Z)$. The topology on $\mathcal{M} \ltimes \mathcal{D}_P^{\theta,\eta}$ is defined by the family of pseudo-metrics $\{\|\cdot\,;\cdot\|_{\theta,\eta;\mathfrak{K}}\}$.

Theorem 3.4 (Theorem 3.10, Lemma 6.7 and Proposition 6.9 of [3])**.** *Let $Z = (\Pi,\Gamma)$ be a model on $\mathcal{T}^{(r)}$. Let W be a sector of $\mathcal{T}^{(r)}$ with regularity $\alpha \le 0$. If $\theta > 0$, $\eta \le \theta$, and $\alpha \wedge \eta > -2$, then there exists a unique continuous linear map $\mathcal{R} : \mathcal{D}_P^{\theta,\eta}(W;Z) \to \mathcal{C}_{\mathfrak{s}}^{\alpha\wedge\eta}$ such that for every compact set $\mathfrak{K} \in \mathbb{R}^2$, we have*

$$|(\mathcal{R}f - \Pi_z f(z))(\rho_z^\delta)| \lesssim \lambda^{\eta-\theta}\delta^\theta \|\Pi\|_{\gamma,\bar{\mathfrak{K}}} \|f\|_{\theta,\eta;\bar{\mathfrak{K}}},$$

uniformly over $f \in \mathcal{D}_P^{\theta,\eta}$, $\rho \in \mathcal{B}_0^2$, $\delta \in (0,1]$, $z \in \mathfrak{K}$, and $\lambda \in (0,1]$ such that $\inf\{|t|\,;(t,x) \in B_{\mathfrak{s}}(z,2\delta)\} \ge \lambda$. Here $\bar{\mathfrak{K}}$ is the 1-fattening of $\mathfrak{K}$. Furthermore, the map $\mathcal{M} \ltimes \mathcal{D}_P^{\theta,\eta}(W) \ni (Z,f) \to \mathcal{R}^Z f \in \mathcal{C}_{\mathfrak{s}}^{\alpha\wedge\eta}$ is locally uniformly continuous.

Let W be a sector of $\mathcal{T}^{(r)}$ such that there exists $\beta \in (0,1)$ such that

$$\langle X^k\,; |k|_{\mathfrak{s}} < r\rangle \subset W, \quad W = \langle \mathbf{1}\rangle \oplus \bigoplus_{l\ge\beta} W_l.$$

Then for every $f \in \mathcal{D}_P^{\theta,\eta}(W)$, $\mathcal{R}f$ coincides with the component of f in T_0. Furthermore we have $\mathcal{R}f \in \mathcal{C}_{\mathfrak{s}}^\beta((0,\infty)\times\mathbb{R})$, and the map $\mathcal{M} \ltimes \mathcal{D}_P^{\theta,\eta}(W) \ni (Z,f) \to \mathcal{R}^Z f \in \mathcal{C}_{\mathfrak{s}}^\beta((0,\infty)\times\mathbb{R})$ is locally uniformly continuous. In fact, for every model Z, $\bar{Z}$, modelled distribution $f \in \mathcal{D}_P(Z)$, $\bar{f} \in \mathcal{D}(\bar{Z})$ and compact set $\mathfrak{K}$ away from P, we have

$$\|\mathcal{R}f - \bar{\mathcal{R}}\bar{f}\|_{\beta,\mathfrak{K}} \lesssim (\inf\{|s|\,; \exists x \in \mathbb{R}\ (s,x) \in \mathfrak{K}\})^{\eta-\theta}(\|f;\bar{f}\|_{\theta,\eta;\bar{\mathfrak{K}}} + \|Z;\bar{Z}\|_{\theta,\bar{\mathfrak{K}}})$$

(Proposition 3.28 of [3] and the definition of modelled distributions).

3.6. **Solution map.** From now on we fix $r > 2$ in Subsection 3.4. We write (1.1) by the mild form:

$$h = G *_{t,x} \{\mathbf{1}_{t>0}((\partial_x h)^2 + \partial_x^\gamma \xi)\} + G *_x h_0,$$

where, G is the heat kernel on $(0,\infty)\times\mathbb{R}$ and h_0 is an initial condition. We reformulate it as an equation of $H \in \mathcal{D}_P^{\theta,\eta}(U)$:

$$H = \mathcal{G}\mathbf{1}_{t>0}((\partial H)^2 + \Xi) + Gh_0. \tag{3.1}$$

Here $Gh_0 = G *_x h_0$ is lifted to an element of $\mathcal{D}_P^{\theta,\eta}(\langle X^k\rangle)$ $(\theta > \eta \vee 0)$ by Lemma 7.5 of [3]. $\mathcal{G}$ is an operator in the following lemma.

Lemma 3.5 (Proposition 6.16, Theorem 7.1 and Lemma 7.3 of [3])**.** *Let $\theta \in (0, r-2)$ and $\eta \in (-2,\theta)$. Assume that $\theta+2, \eta+2 \notin \mathbb{N}$. Then, for each admissible and periodic model Z on $\mathcal{T}^{(r)}$, there exists a continuous linear map $\mathcal{G} : \mathcal{D}_P^{\theta,\eta}(V) \to \mathcal{D}_P^{\bar{\theta},\bar{\eta}}(U)$ $(\bar{\theta} = \theta+2,\ \bar{\eta} = \eta\wedge\alpha_0+2)$ such that for every $f \in \mathcal{D}_P^{\theta,\eta}(V)$ we have*

(1) *$\mathcal{G}f - \mathcal{I}f$ takes values in $\langle X^k\rangle$.*

(2) $\mathcal{RG}f = G * \mathcal{R}f$.

The nonlinearlity is naturally extended to $\mathcal{D}_P^{\theta,\eta}$.

Lemma 3.6 (Propositions 6.12 and 6.15 of [3])**.** *For every $f \in \mathcal{D}_P^{\theta,\eta}(U)$ with $\theta > 1$ and $\eta < \alpha_0 + 2$, we have $(\partial f)^2 + \Xi \in \mathcal{D}_P^{\theta+\alpha_0,\eta+\alpha_0}(V)$. Furthermore, the map $f \mapsto (\partial f)^2 + \Xi$ is locally Lipschitz continuous.*

proof. From Proposition 6.15 of [3], $\partial f \in \mathcal{D}_P^{\theta-1,\eta-1}$. Since the regularity of the sector where ∂f takes value is $|\mathcal{I}'(\Xi)|_{\mathfrak{s}} = \alpha_0 + 1$, we have $(\partial f)^2 + \Xi \in \mathcal{D}_P^{\bar{\theta},\bar{\eta}}$ with $\bar{\theta} = (\theta-1)+(\alpha_0+1) = \theta+\alpha_0$ and $\bar{\eta} = (\eta+\alpha_0)\wedge(2\alpha_0+2) = \eta+\alpha_0$, from Proposition 6.15 of [3]. Lipschitz continuity is also obtained from these propositions. $\square$

Now we have the continuity of the solution map.

Theorem 3.7 (Theorem 7.8 of [3])**.** *Let $0 \le \gamma < \frac{1}{2}$, and $-2 < \alpha_0 < -\frac{3}{2} - \gamma$. Let $\theta \in (-\alpha_0, r - \alpha_0 - 2)$, and $\eta \in (0, \alpha_0 + 2)$. Then, for every periodic $h_0 \in \mathcal{C}^\eta(\mathbb{R})$ and admissible and periodic model Z, there exists a time $T = T(h_0, Z) > 0$ such that, for every $t < T$ there exists a unique solution $H \in \mathcal{D}_P^{\theta,\eta}(U)$ to (3.1) on $[0,t]$, and it holds that $T = \infty$ or $\lim_{t\to\infty} \|\mathcal{R}H(t,\cdot)\|_\eta = \infty$. Furthermore, the solution map $\mathcal{S} : (h_0, Z) \mapsto H$ is jointly uniformly continuous in a neighborhood around (h_0, Z).*

4. Renormalization

For each $\epsilon > 0$, we can lift the noise $\partial_x^\gamma \xi_\epsilon$ to a canonical model $Z^{(\epsilon)} = (\Pi^{(\epsilon)}, \Gamma^{(\epsilon)})$ on $\mathcal{T}^{(r)}$. We define the linear map $\mathbf{\Pi}^{(\epsilon)} : T_r^- \to C(\mathbb{R}^2)$ by

$$(\mathbf{\Pi}^{(\epsilon)}\mathbf{1})(z) = 1, \quad (\mathbf{\Pi}^{(\epsilon)}X_0)(z) = t, \quad (\mathbf{\Pi}^{(\epsilon)}X_1)(z) = x, \quad (\mathbf{\Pi}^{(\epsilon)}\Xi)(z) = \partial_x^\gamma \xi_\epsilon(z),$$
$$(\mathbf{\Pi}^{(\epsilon)}\tau\bar{\tau})(z) = (\mathbf{\Pi}^{(\epsilon)}\tau)(z)(\mathbf{\Pi}^{(\epsilon)}\bar{\tau})(z), \quad (\mathbf{\Pi}^{(\epsilon)}\mathcal{I}_k\tau)(z) = \partial^k K * (\mathbf{\Pi}^{(\epsilon)}\tau)(z).$$

We define the family $\{f_z^{(\epsilon)} \in G\,; z \in \mathbb{R}^2\}$ by

$$f_z^{(\epsilon)}(\mathbf{1}) = 1, \quad f_z^{(\epsilon)}(X_0) = -t, \quad f_z^{(\epsilon)}(X_1) = -x,$$
$$f_z^{(\epsilon)}(\mathcal{J}_k\tau) = -\partial^k K * (\mathbf{\Pi}^{(\epsilon)}\Gamma_{f_z^{(\epsilon)}}\tau)(z) \quad (|\tau|_{\mathfrak{s}} + 2 - |k|_{\mathfrak{s}} > 0).$$

Lemma 4.1. *Let $\Pi_z^{(\epsilon)} = \mathbf{\Pi}^{(\epsilon)}\Gamma_{f_z^{(\epsilon)}}$ and $\Gamma_{z\bar{z}}^{(\epsilon)} = (\Gamma_{f_z^{(\epsilon)}})^{-1}\Gamma_{f_{\bar{z}}^{(\epsilon)}}$. Then $Z^{(\epsilon)} = (\Pi^{(\epsilon)}, \Gamma^{(\epsilon)})$ is an admissible and periodic model.*

4.1. **Renormalization procedure.** We introduce a renormalization of $Z^{(\epsilon)}$ following Section 8.3 of [3]. Let $\mathcal{F}_0 \subset \mathcal{F}$ be a subset such that

(1) $\{\tau \in \mathcal{F}\,; |\tau|_{\mathfrak{s}} \le 0\} \subset \mathcal{F}_0$.
(2) There exists a subset $\mathcal{F}_* \subset \mathcal{F}_0$ such that $\Delta\mathcal{F}_0 \subset \langle\mathcal{F}_0\rangle \otimes \langle\mathcal{F}_0^+\rangle$, where $\mathcal{F}_0^+$ is the minimal subset of $\mathcal{F}^+$ such that

$$(1)\ \mathbf{1}, X_0, X_1 \in \mathcal{F}_0^+, \quad (2)\ \tau, \bar{\tau} \in \mathcal{F}_0^+ \Rightarrow \tau\bar{\tau} \in \mathcal{F}_0^+,$$
$$(3)\ \tau \in \mathcal{F}_* \setminus \{X^l\,; l \in \mathbb{Z}_+^2\}, k \in \mathbb{Z}_+^2, |\tau|_{\mathfrak{s}} + 2 - |k|_{\mathfrak{s}} > 0 \Rightarrow \mathcal{J}_k\tau \in \mathcal{F}_0^+.$$

We write $\mathcal{H}_0 = \langle\mathcal{F}_0\rangle$ and $\mathcal{H}_0^+ = \langle\mathcal{F}_0^+\rangle$.

Let $M : \mathcal{H}_0 \to \mathcal{H}_0$ be a linear map such that $M\mathcal{I}_k\tau = \mathcal{I}_k M\tau$ for every $\tau \in \mathcal{F}_0$ such that $\mathcal{I}_k\tau \in \mathcal{F}_0$, and $MX^k = X^k$. Then two linear maps $\hat{M} : \mathcal{H}_0^+ \to \mathcal{H}_0^+$ and $\Delta^M : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{H}_0^+$ are uniquely determined by the following properties.

(1) $\hat{M}\mathcal{J}_k\tau = \mathcal{M}(\mathcal{J}_k \otimes \mathrm{id})\Delta^M\tau$, $\hat{M}X^k = X^k$,

(2) $\hat{M}(\tau\bar{\tau}) = (\hat{M}\tau)(\hat{M}\bar{\tau})$,
(3) $(\mathrm{id}\otimes\mathcal{M})(\Delta\otimes\mathrm{id})\Delta^M\tau = (M\otimes\hat{M})\Delta\tau$

(Proposition 8.36 of [3]). Furthermore, the linear map $\hat{\Delta}^M : \mathcal{H}_0^+ \to \mathcal{H}_0^+\otimes\mathcal{H}_0^+$ is defined by

$$(\mathcal{A}\hat{M}\mathcal{A}\otimes\hat{M})\Delta^+ = (\mathrm{id}\otimes\mathcal{M})(\Delta^+\otimes\mathrm{id})\hat{\Delta}^M,$$

since $(\mathrm{id}\otimes\mathcal{M})(\Delta^+\otimes\mathrm{id})$ is invertible on $\mathcal{H}_0^+\otimes\mathcal{H}_0^+$ from the definition of Δ^+.

Lemma 4.2 (Theorem 8.44 of [3]). *Let $\mathcal{F}_0$ and M as above. Assume that for every $\tau\in\mathcal{F}_0$ and $\hat{\tau}\in\mathcal{F}_0^+$ we can write*

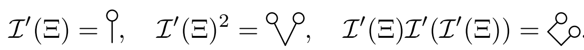
$$\Delta^M\tau = \tau\otimes\mathbf{1} + \sum_{|\tau^{(1)}|_\mathfrak{s}>|\tau|_\mathfrak{s}} \tau^{(1)}\otimes\tau^{(2)}, \quad \hat{\Delta}^M\hat{\tau} = \hat{\tau}\otimes\mathbf{1} + \sum_{|\hat{\tau}^{(1)}|_\mathfrak{s}>|\hat{\tau}|_\mathfrak{s}} \hat{\tau}^{(1)}\otimes\hat{\tau}^{(2)}.$$

Then for every admissible model $(\mathbf{\Pi}, f)$ on $\mathcal{T}^{(r)}$, the maps $\mathbf{\Pi}^M : \mathcal{H}_0 \to \mathcal{S}'(\mathbb{R}^2)$ and $f_z^M : \mathcal{H}_0^+ \to \mathbb{R}$ defined by

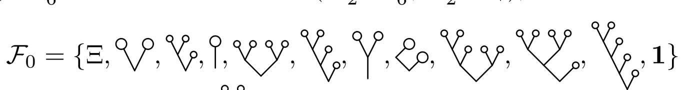
$$\mathbf{\Pi}^M = \mathbf{\Pi}M, \quad f_z^M = f_z\hat{M}$$

are uniquely extended to an admissible model $Z^M = (\Pi^M, \Gamma^M)$ on $\mathcal{T}^{(r)}$.

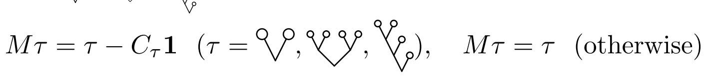
4.2. **Renormalization map.** We construct renormalization maps when $0\le\gamma<\frac{1}{4}$. We write each element of $\mathcal{F}_0$ as a graph with one root. We draw a circle to represent Ξ. In order to represent $\mathcal{I}'(\tau)$, we draw a downward line starting at the root of τ. Then the root of $\mathcal{I}'(\tau)$ is another vertex of this downward line, which is not the root of τ. In order to represent $\tau\bar{\tau}$, we joint the trees τ and $\bar{\tau}$ at their roots. For example,



4.2.1. $0\le\gamma<\frac{1}{6}$. Provided that $\alpha_0\in(-\frac{3}{2}-\frac{1}{6}, -\frac{3}{2}-\gamma)$, it is sufficient to set



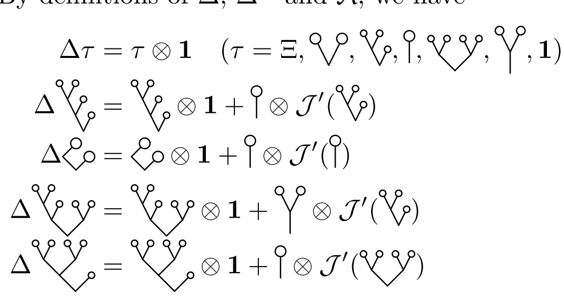
For constants , we define a linear map $M : \mathcal{H}_0\to\mathcal{H}_0$ by



Lemma 4.3. *M satisfies the condition of Lemma 4.2. For every $\tau\in\mathcal{F}_0$, the renormalized model $Z^{(\epsilon),M} = (Z^{(\epsilon)})^M$ satisfies*

$$\Pi_z^{(\epsilon),M}\tau = \Pi_z^{(\epsilon)}M\tau.$$

proof. By definitions of Δ, Δ^+ and $\mathcal{A}$, we have



$$\Delta \; = \; \otimes \mathbf{1} + \; \otimes \mathcal{J}'(\;) + \; \otimes \mathcal{J}'(\;),$$
$$\Delta^+ \mathcal{J}_k(\tau) = \sum_l \mathcal{J}_{k+l}(\tau) \otimes \frac{(-X)^l}{l!} + \mathbf{1} \otimes \mathcal{J}_k(\tau) \quad (\tau = \;,\;,\;),$$
$$\Delta^+ \mathcal{J}_k(\;) = \sum_l \mathcal{J}_{k+l}(\;) \otimes \frac{(-X)^l}{l!} + \sum_l \mathcal{J}_{k+l}(\;) \otimes \frac{(-X)^l}{l!} \mathcal{J}'(\;)$$
$$+ \mathbf{1} \otimes \mathcal{J}_k(\;),$$
$$\mathcal{A}\mathcal{J}_k(\tau) = -\sum_l \frac{X^l}{l!} \mathcal{J}_{k+l}(\tau) \quad (\tau = \;,\;,\;),$$
$$\mathcal{A}\mathcal{J}_k(\;) = -\sum_l \frac{X^l}{l!} \mathcal{J}_{k+l}(\;) + \sum_l \frac{X^l}{l!} \mathcal{J}_{k+l}(\;)\mathcal{J}'(\;).$$

First we verify that

$$\Delta^M \tau = M\tau \otimes \mathbf{1} \quad (\tau \in \mathcal{F}_0), \qquad \hat{M}\mathcal{J}_k \tau = \mathcal{J}_k \tau \quad (\tau \in \mathcal{F}_*,\ k \in \mathbb{Z}_+^2). \tag{4.1}$$

For each $\tau \in \mathcal{F}_0$, we can write $M\tau = \tau - C_\tau \mathbf{1}$ with some constant C_τ. For $\tau = \Xi$, , , , , , $\mathbf{1}$, we have $\Delta\tau = \tau \otimes \mathbf{1}$ so that

$$(M \otimes \hat{M})\Delta\tau = (M \otimes \hat{M})(\tau \otimes \mathbf{1}) = M\tau \otimes \mathbf{1},$$
$$\begin{aligned}(\mathrm{id} \otimes \mathcal{M})(\Delta \otimes \mathrm{id})(M\tau \otimes \mathbf{1}) &= (\mathrm{id} \otimes \mathcal{M})(\tau \otimes \mathbf{1} \otimes \mathbf{1} - C_\tau \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}) \\ &= \tau \otimes \mathbf{1} - C_\tau \mathbf{1} \otimes \mathbf{1} = M\tau \otimes \mathbf{1}.\end{aligned}$$

This implies $\Delta^M \tau = M\tau \otimes \mathbf{1}$. For $\tau = \;,\;,\;$, we also have

$$\hat{M}\mathcal{J}_k(\tau) = \mathcal{M}(\mathcal{J}_k \otimes \mathrm{id})\Delta^M \tau = \mathcal{M}(\mathcal{J}_k \otimes \mathrm{id})(\tau \otimes \mathbf{1} - C_\tau \mathbf{1} \otimes \mathbf{1}) = \mathcal{J}_k(\tau).$$

For $\tau = \;,\;,\;$, we have $\Delta\tau = \tau \otimes \mathbf{1} + \mathcal{J}'(\bar{\tau})$ with $\bar{\tau} = \;,\;,\;$ respectively, so that

$$(M \otimes \hat{M})\Delta\tau = (M \otimes \hat{M})(\tau \otimes \mathbf{1} + \; \otimes \mathcal{J}'(\bar{\tau})) = M\tau \otimes \mathbf{1} + \; \otimes \mathcal{J}'(\bar{\tau}),$$
$$\begin{aligned}(\mathrm{id} \otimes \mathcal{M})(\Delta \otimes \mathrm{id})(M\tau \otimes \mathbf{1}) &= (\mathrm{id} \otimes \mathcal{M})(\Delta \otimes \mathrm{id})(\tau \otimes \mathbf{1} - C_\tau \mathbf{1} \otimes \mathbf{1}) \\ &= \tau \otimes \mathbf{1} + \; \otimes \mathcal{J}'(\bar{\tau}) - C_\tau \mathbf{1} \otimes \mathbf{1} \\ &= M\tau \otimes \mathbf{1} + \; \otimes \mathcal{J}'(\bar{\tau}).\end{aligned}$$

This implies $\Delta^M \tau = M\tau \otimes \mathbf{1}$. Similarly to above, we also have

$$\hat{M}\mathcal{J}_k(\;) = \mathcal{M}(\mathcal{J}_k \otimes \mathrm{id})\Delta^M \; = \mathcal{M}(\mathcal{J}_k \otimes \mathrm{id})(\; \otimes \mathbf{1} - C\, \mathbf{1} \otimes \mathbf{1}) = \mathcal{J}_k(\;).$$

For $\tau = \;$, we have

$$\begin{aligned}(M \otimes \hat{M})\Delta\tau &= (M \otimes \hat{M})(\tau \otimes \mathbf{1} + \; \otimes \mathcal{J}'(\;) + \; \otimes \mathcal{J}'(\;)) \\ &= \tau \otimes \mathbf{1} + \; \otimes \mathcal{J}'(\;) + \; \otimes \mathcal{J}'(\;),\end{aligned}$$
$$\begin{aligned}(\mathrm{id} \otimes \mathcal{M})(\Delta \otimes \mathrm{id})(M\tau \otimes \mathbf{1}) &= (\mathrm{id} \otimes \mathcal{M})(\Delta \otimes \mathrm{id})(\tau \otimes \mathbf{1}) \\ &= \tau \otimes \mathbf{1} + \; \otimes \mathcal{J}'(\;) + \; \otimes \mathcal{J}'(\;).\end{aligned}$$

This implies $\Delta^M \tau = M\tau \otimes \mathbf{1}$.

Next we verify that

$$\hat{\Delta}^M \mathcal{J}_k \tau = \mathcal{J}_k \tau \otimes \mathbf{1} \quad (\tau \in \mathcal{F}_*,\ k \in \mathbb{Z}_+^2). \tag{4.2}$$

From (4.1), for $\tau = \ , \ , \ $ we have

$$\begin{aligned}
\mathcal{A}\hat{M}\mathcal{A}\mathcal{J}_k(\tau) &= -\sum_l \mathcal{A}\hat{M}\left(\frac{X^l}{l!}\mathcal{J}_{k+l}(\tau)\right) = -\sum_l \mathcal{A}\left(\frac{X^l}{l!}\mathcal{J}_{k+l}(\tau)\right)\\
&= -\sum_{l,m}\frac{(-X)^l}{l!}\frac{X^m}{m!}\mathcal{J}_{k+l+m}(\tau) = \mathcal{J}_k(\tau).
\end{aligned}$$

For $\ $, similarly we have

$$\begin{aligned}
\mathcal{A}\hat{M}\mathcal{A}\mathcal{J}_k(\ ) &= -\sum_l \mathcal{A}\hat{M}\left(\frac{X^l}{l!}\mathcal{J}_{k+l}(\ )\right) + \sum_l \mathcal{A}\hat{M}\left(\frac{X^l}{l!}\mathcal{J}_{k+l}(\ )\mathcal{J}'(\ )\right)\\
&= -\sum_l \mathcal{A}\left(\frac{X^l}{l!}\mathcal{J}_{k+l}(\ )\right) + \sum_l \mathcal{A}\left(\frac{X^l}{l!}\mathcal{J}_{k+l}(\ )\mathcal{J}'(\ )\right)\\
&= \sum_{l,m}\frac{(-X)^l}{l!}\frac{X^m}{m!}\mathcal{J}_{k+l+m}(\ ) - \sum_{l,m}\frac{(-X)^l}{l!}\frac{X^m}{m!}\mathcal{J}_{k+l+m}(\ )\mathcal{J}'(\ )\\
&\qquad + \sum_{l,m}\frac{(-X)^l}{l!}\frac{X^m}{m!}\mathcal{J}_{k+l+m}(\ )\mathcal{J}'(\ )\\
&= \mathcal{J}_k(\ ).
\end{aligned}$$

From these identities, we have

$$(\mathcal{A}\hat{M}\mathcal{A}\otimes\hat{M})\Delta^+\mathcal{J}_k(\tau) = \Delta^+\mathcal{J}_k(\tau) \quad (\tau \in \mathcal{F}_*,\ k \in \mathbb{Z}_+^2).$$

Since $(\mathrm{id}\otimes\mathcal{M})(\Delta^+\otimes\mathrm{id})(\mathcal{J}_k(\tau)\otimes\mathbf{1}) = \Delta^+\mathcal{J}_k(\tau)$, we obtain (4.2).

From (4.2), we can apply Lemma 4.2 and obtain the renormalized model $Z^{(\epsilon),M}$. From (4.1), we have

$$f_z^{(\epsilon),M}(\mathcal{J}_k\tau) = f_z^{(\epsilon)}(\hat{M}\mathcal{J}_k\tau) = f_z^{(\epsilon)}(\mathcal{J}_k\tau) \quad (\tau \in \mathcal{F}_*,\ k \in \mathbb{Z}_+^2).$$

This implies that $\Gamma_{f_z^{(\epsilon),M}} = \Gamma_{f_z^{(\epsilon)}}$ on $\mathcal{H}_0$. Since $(\Gamma_{f_z^{(\epsilon)}} - \mathrm{id})\mathcal{F}_0 \subset \langle\ ,\ ,\ \rangle$, for every $\tau \in \mathcal{F}_0$ we have

$$\Gamma_{f_z^{(\epsilon)}}\tau - \tau = M(\Gamma_{f_z^{(\epsilon)}}\tau - \tau) = M\Gamma_{f_z^{(\epsilon)}}\tau - \tau + C_\tau\mathbf{1},$$

so that $M\Gamma_{f_z^{(\epsilon)}}\tau = \Gamma_{f_z^{(\epsilon)}}\tau - C_\tau\mathbf{1} = \Gamma_{f_z^{(\epsilon)}}M\tau$. Therefore, for every $\tau \in \mathcal{F}_0$ we have

$$\Pi_z^{(\epsilon),M}\tau = \mathbf{\Pi}^{(\epsilon),M}\Gamma_{f_z^{(\epsilon),M}}\tau = \mathbf{\Pi}^{(\epsilon)}M\Gamma_{f_z^{(\epsilon)}}\tau = \mathbf{\Pi}^{(\epsilon)}\Gamma_{f_z^{(\epsilon)}}M\tau = \Pi_z^{(\epsilon)}M\tau.$$

□

Proposition 4.4. *Let $\mathcal{S} : (h_0, Z) \mapsto H$ be the solution map defined in Theorem 3.7. Then $h_\epsilon^M = \mathcal{R}\mathcal{S}(h_0, Z^{(\epsilon),M})$ solves the equation*

$$\partial_t h_\epsilon^M = \partial_x^2 h_\epsilon^M + (\partial_x h_\epsilon^M)^2 - (C_{\ } + C_{\ } + 4C_{\ }) + \partial_x^\gamma \xi_\epsilon \tag{4.3}$$

with the initial condition h_0.

proof. By the definition of $\mathcal{T}^{(r)}$, every solution $H \in \mathcal{D}_P^{\theta,\eta}(U)$ to (3.1) with $\theta > -\alpha_0$ has the following form.

$$H = h\mathbf{1} + \mathcal{I}(\Xi) + \mathcal{I}(\;) + h'X_1 + 2\mathcal{I}(\;) + 2h'\mathcal{I}(\;) + \mathcal{I}(\;) + 4\mathcal{I}(\;).$$

Here h and h' are functions on $\mathbb{R}^2$. Then ∂H and $(\partial H)^2 + \Xi$ are given by

$$\partial H = \; + \; + h'\mathbf{1} + 2\mathcal{I}'(\;) + 2h'\mathcal{I}'(\;) + \mathcal{I}'(\;) + 4\mathcal{I}'(\;),$$

$$(\partial H)^2 + \Xi = \Xi + \; + 2\; + 2h'\; + \; + 4\; + 2h'\; + 4h'\; + 4\; + 2\; + 8\;.$$

From these representations, we obtain

$$M(\partial H) = \partial H, \quad M((\partial H)^2 + \Xi) = (\partial H)^2 + \Xi - (C + C + 4C).$$

Hence we have

$$M((\partial H)^2 + \Xi) = (M\partial H)^2 + \Xi - (C + C + 4C).$$

From Remark 3.15 of [3] and Lemma 4.3, for the reconstruction operator $\mathcal{R}$ of $\hat{Z}^{(\epsilon),M}$ we have

$$\mathcal{R}((\partial H)^2 + \Xi) = (\partial_x \mathcal{R}H)^2 + \partial_x^\gamma \xi_\epsilon - (C + C + 4C).$$

Let $H = \mathcal{S}(h_0, Z^{(\epsilon),M})$. Applying $\mathcal{R}^{(\epsilon),M}$ to both sides of (3.1), we have

$$\mathcal{R}H = G * \{\mathbf{1}_{t>0}((\partial_x \mathcal{R}H)^2 + \partial_x^\gamma \xi_\epsilon - (C + C + 4C))\} + Gh_0.$$

Hence $h_\epsilon^M = \mathcal{R}H$ solves (4.3). $\square$

The goal of this paper is the following theorem.

Theorem 4.5. *Let $C_\tau^{(\epsilon)}$ ($\tau =$, ,) be constants defined in Section 5. Let $\hat{Z}^{(\epsilon)} = Z^{(\epsilon),M^{(\epsilon)}}$ be the corresponding renormalized model. Then there exists an admissible and periodic random model $\hat{Z}$ and $\kappa > 0$, such that for every $\zeta > 0$, compact set $\mathfrak{K}$ and $p \geq 1$, we have the bounds*

$$\mathbb{E}\|\hat{Z}\|_{\zeta;\mathfrak{K}}^p \lesssim 1, \quad \mathbb{E}\|\hat{Z}^{(\epsilon)}; \hat{Z}\|_{\zeta;\mathfrak{K}}^p \lesssim \epsilon^{\kappa p}.$$

4.2.2. $\frac{1}{6} \leq \gamma < \frac{1}{4}$. Provided $\alpha_0 \in (-\frac{3}{2} - \frac{1}{4}, -\frac{3}{2} - \gamma)$, it is sufficient to set

$$\mathcal{F}_0 = \{\Xi, \ldots, \mathbf{1}\},$$

$$\mathcal{F}_* = \{\ldots\}.$$

Set $\mathcal{F}_{\text{re}} = \{\ldots\}$. For constants C_τ ($\tau \in \mathcal{F}_{\text{re}}$), we define a linear map $M : \mathcal{H}_0 \to \mathcal{H}_0$ by

$$M\tau = \tau - C_\tau \quad (\tau \in \mathcal{F}_{\text{re}}), \quad M\tau = \tau \quad (\text{otherwise}).$$

The following results are obtained similarly to Lemma 4.3 and Proposition 4.4.

Lemma 4.6. *M satisfies the condition of Lemma 4.2. For every $\tau \in \mathcal{F}_0$, the renormalized model $Z^{(\epsilon),M}$ satisfies*

$$\Pi_z^{(\epsilon),M}\tau = \Pi_z^{(\epsilon)} M\tau.$$

Proposition 4.7. *$h_\epsilon^M = \mathcal{RS}(h_0, Z^{(\epsilon),M})$ solves the equation*

$$\partial_t h_\epsilon^M = \partial_x^2 h_\epsilon^M + (\partial_x h_\epsilon^M)^2 - (C + C + 4C + 4C + 2C + 8C + 8C + 4C + 16C) + \partial_x^\gamma \xi_\epsilon$$

with the initial condition h_0.

The goal of this paper is the following theorem.

Theorem 4.8. *Let $C_\tau^{(\epsilon)}$ ($\tau \in \mathcal{F}_{\mathrm{re}}$) be constants defined in Section 7. Let $\hat{Z}^{(\epsilon)} = Z^{(\epsilon),M^{(\epsilon)}}$ be the corresponding renormalized model. Then there exists an admissible and periodic random model $\hat{Z}$ and $\kappa > 0$, such that for every $\zeta > 0$, compact set $\mathfrak{K}$ and $p \geq 1$, we have the bounds*

$$\mathbb{E}\|\hat{Z}\|_{\zeta;\mathfrak{K}}^p \lesssim 1, \quad \mathbb{E}\|\hat{Z}^{(\epsilon)}; \hat{Z}\|_{\zeta;\mathfrak{K}}^p \lesssim \epsilon^{\kappa p}.$$

4.3. **Wiener chaos expansion.** The processes $\Pi_z\tau$ belong to Wiener chaos constructed from the space-time white noise $\{W_h\}$ on $\mathbb{R}\times\mathbb{T}$. We denote by $I_1 : H = L^2(\mathbb{R}\times\mathbb{T}) \to L^2(\Omega,\mathbb{P})$ a linear isometry defined by $I_1(h) = W_h$. We also denote by $I_k : H^{\otimes k}(\simeq L^2((\mathbb{R}\times\mathbb{T})^k)) \to L^2(\Omega,\mathbb{P})$ the k-th stochastic integral defined in Theorem 7.25 of [6].

For each $\tau \in \mathcal{F}$, we can define the number $\|\tau\|$ by

$$\|\mathbf{1}\| = \|X_0\| = \|X_1\| = 0, \quad \|\Xi\| = 1, \quad \|\tau\bar{\tau}\| = \|\tau\| + \|\bar{\tau}\|, \quad \|\mathcal{I}_k\tau\| = \|\tau\|.$$

Lemma 4.9. *For each $\tau \in \mathcal{F}_- := \{\tau \in \mathcal{F}\,; |\tau|_\mathfrak{s} \leq 0\}$ and $z \in \mathbb{R}^2$, there exists a family of functions $\{\mathcal{W}^{(\epsilon,k)}(\tau)(z) = \mathcal{W}^{(\epsilon,k)}(\tau)(z; w_1,\dots,w_k) \in H^{\otimes k}\}_{k\in\|\tau\|-2\mathbb{Z}_+, k\geq 0}$ such that*

$$(\Pi_0^{(\epsilon)}\tau)(z) = \sum_k I_k(\mathcal{W}^{(\epsilon,k)}(\tau)(z)). \tag{4.4}$$

proof. For Ξ, by the definition of $\partial_x^\gamma\xi_\epsilon$ we have

$$\begin{aligned}(\Pi_0^{(\epsilon)}\Xi)(z) &= \partial_x^\gamma\xi_\epsilon(z) = \partial_x^\gamma\xi(\rho_\epsilon(z-\cdot)) = \partial_x^\gamma W(\pi\rho_\epsilon(z-\cdot)) \\ &= W(\partial_x^\gamma(\mathbb{T})\pi\rho_\epsilon(z-\cdot)) = W(\pi\partial_x^\gamma(\mathbb{R})\rho_\epsilon(z-\cdot)).\end{aligned}$$

Here $\partial_x^\gamma(\mathbb{T})$ and $\partial_x^\gamma(\mathbb{R})$ denote the fractional derivatives defined in $\mathbb{T}$ and $\mathbb{R}$, respectively. Hence

$$(\Pi_0^{(\epsilon)}\Xi)(z) = I_1(\mathcal{W}^{(\epsilon,1)}(\Xi)(z)), \quad \mathcal{W}^{(\epsilon,1)}(\Xi)(z;w) = \pi\partial_x^\gamma\rho_\epsilon(z-w).$$

If τ satisfies (4.4), it is clear that $\mathcal{I}\tau$ and $\mathcal{I}'\tau$ also. Regarding the product of two variables, we can use Theorem 7.33 of [6]. $\square$

In order to prove Theorems 4.5 and 4.8, it is sufficient to obtain estimates in the following theorem.

Theorem 4.10 (Theorem 10.7 of [3])**.** *For the renormalized model $\hat{Z}^{(\epsilon)}$ in Theorems 4.5 and 4.8, assume that there exist $\kappa > 0$ and $\iota > 0$, such that for every $\varphi \in \mathcal{B}_0^2$ and*

$\tau \in \mathcal{F}_-$ there exists a random variable $(\hat{\Pi}_0\tau)(\varphi)$ belonging to the inhomogeneous Wiener chaos of order $\|\tau\|$ such that

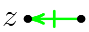
$$\mathbb{E}|(\hat{\Pi}_0\tau)(\varphi_0^\lambda)|^2 \lesssim \lambda^{2|\tau|_\mathfrak{s}+\iota}, \quad \mathbb{E}|(\hat{\Pi}_0\tau - \hat{\Pi}_0^{(\epsilon)}\tau)(\varphi_0^\lambda)|^2 \lesssim \epsilon^{2\kappa}\lambda^{2|\tau|_\mathfrak{s}+\iota}.$$

uniformly over $0 < \lambda < C$. Then there exists a unique admissible and periodic random model $\hat{Z}$, such that for every $\zeta > 0$, compact set $\mathfrak{K} \subset \mathbb{R}^2$ and $p \geq 1$ we have

$$\mathbb{E}\|\hat{Z}\|_{\zeta;\mathfrak{K}}^p \lesssim 1, \quad \mathbb{E}\|\hat{Z};\hat{Z}^{(\epsilon)}\|_{\zeta;\mathfrak{K}}^p \lesssim \epsilon^{\kappa p}.$$

4.4. **Graphical notations.** In order to write kernels $\mathcal{W}(\tau)$, graphical notations as in Section 10.5 of [3] are useful. Each kernel is written as a directed graph which may contain nondirected edges. One vertex ($\bullet$) represents a variable in $\mathbb{R}^2$. When a vertex is written by orange ($\bullet$), it is integrated out. One edge represents a function of two variables at its vertexes. We use special (directed or nondirected) edges as follows.

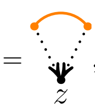
z $w = \pi\partial_x^\gamma K'_\epsilon(z-w)$, $\qquad$ z  $w = \pi\partial_x^\gamma K'(z-w)$,

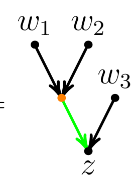
z $w = K'(z-w)$, $\qquad$ z  $w = K'(z-w) - K'(-w)$,

z $w = \mathbf{1}_{z,w\in\mathbb{R}\times[0,1)}\delta(z-w)$.

Here we write $K'_\epsilon = K' * \rho_\epsilon$

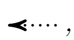
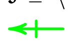
For example, we write

$$\int_{w\in\mathbb{R}\times[0,1)} (\pi\partial_x^\gamma K'_\epsilon(z-w))^2 dw =$$  ,

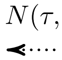
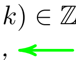
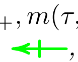
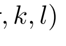
$$\int K'(z-u)\pi\partial_x^\gamma K'(u-w_1)\pi\partial_x^\gamma K'(u-w_2)\pi\partial_x^\gamma K'(z-w_3)du =$$  .

For each $\tau \in \mathcal{F}_-$, the kernel $\mathcal{W}^{(\epsilon,\|\tau\|)}(\tau)(z;w_1,\dots,w_{\|\tau\|})$ is given recursively by

$$\begin{aligned}
&\mathcal{W}^{(\epsilon,1)}(\Xi)(z;w) = \pi\partial_x^\gamma\rho_\epsilon(z-w),\\
&\mathcal{W}^{(\epsilon,\|\tau\bar{\tau}\|)}(\tau\bar{\tau})(z;w_1,\dots,w_{\|\tau\bar{\tau}\|})\\
&\qquad = \mathcal{W}^{(\epsilon,\|\tau\|)}(\tau)(z;w_1,\dots,w_{\|\tau\|})\mathcal{W}^{(\epsilon,\|\bar{\tau}\|)}(\bar{\tau})(z;w_{\|\tau\|+1},\dots,w_{\|\tau\bar{\tau}\|}),\\
&\mathcal{W}^{(\epsilon,\|\mathcal{I}'\tau\|)}(\mathcal{I}'\tau)(z;w_1,\dots,w_{\|\mathcal{I}'\tau\|})\\
&\qquad = \int\Big(K'(z-u) - \sum_{|k|_\mathfrak{s}<|\tau|_\mathfrak{s}+1} z^k\partial^k K'(-u)\Big)\mathcal{W}^{(\epsilon,\|\tau\|)}(\tau)(u;w_1,\dots,w_{\|\tau\|})du.
\end{aligned}$$

Lemma 4.11. *For each $\tau \in \mathcal{F}_- \setminus \{\Xi, \mathbf{1}\}$, the kernel $\mathcal{W}^{(\epsilon,\|\tau\|)}(\tau)$ is written as a graph using only*  , ,  *for its edges. Furthermore, kernels $\mathcal{W}^{(\epsilon,k)}(\tau)$ with $k < \|\tau\|$ can be decomposed into*

$$\mathcal{W}^{(\epsilon,k)}(\tau)(z) = \sum_{l=1}^{N(\tau,k)} m(\tau,k,l)\mathcal{W}_l^{(\epsilon,k)}(\tau)(z)$$

with $N(\tau,k) \in \mathbb{Z}_+, m(\tau,k,l) \in \mathbb{Z}$, and each $\mathcal{W}_l^{(\epsilon,k)}(\tau)$ can be written as graphs using only  ,  ,  ,  *for their edges.*

proof. For , we have

$$\begin{aligned}\mathcal{W}^{(\epsilon,1)}()(z;w) &= K' * \pi\partial_x^\gamma\rho_\epsilon(z-w) = \pi(K' * \partial_x^\gamma\rho_\epsilon)(z-w) = \pi\partial_x^\gamma K'_\epsilon(z-w)\\ &= z \bullet\!\!\leftarrow\cdots\bullet\, w\,.\end{aligned}$$

For $\tau,\bar\tau \in \mathcal{F}$ such that $\mathcal{W}^{(\epsilon,\|\tau\|)}(\tau)$ and $\mathcal{W}^{(\epsilon,\|\bar\tau\|)}(\bar\tau)$ are written by using only $\leftarrow\cdots$, $\longleftarrow$, $\leftarrow\!\!+\!\!-$ for their edges, the product $\mathcal{W}^{(\epsilon,\|\tau\bar\tau\|)}(\tau\bar\tau)$ also from its definition.

We note that a factor $\mathcal{I}'(\bar\tau)$ with $|\bar\tau|_{\mathfrak{s}} > 0$ cannot appear in any $\tau \in \mathcal{F}_-$ since the smallest homogeneity of elements containing this factor is $|\mathcal{I}'(\Xi)\mathcal{I}'(\bar\tau)|_{\mathfrak{s}} = \alpha_0 + 2 + |\bar\tau|_{\mathfrak{s}} > 0$. Hence it is sufficient to consider a factor $\mathcal{I}'(\bar\tau)$ with $|\bar\tau|_{\mathfrak{s}} \le 0$. If $\mathcal{W}^{(\epsilon,\|\bar\tau\|)}$ are written by using only $\leftarrow\cdots$, $\longleftarrow$, $\leftarrow\!\!+\!\!-$ for its edges, $\mathcal{I}'(\bar\tau)$ also because

$$K'(z-u) \quad (|\tau|_{\mathfrak{s}}+1 \le 0), \qquad K'(z-u) - K'(-u) \quad (|\tau|_{\mathfrak{s}}+1 > 0)$$

are written by $\longleftarrow$, $\leftarrow\!\!+\!\!-$.

Kernels $\mathcal{W}_l^{(\epsilon,k)}(\tau)(z)$ with $k < \|\tau\|$ are obtained by contractions of $\mathcal{W}^{(\epsilon,\|\tau\|)}(\tau)(z)$ form Theorem 7.33 of [6], i.e.

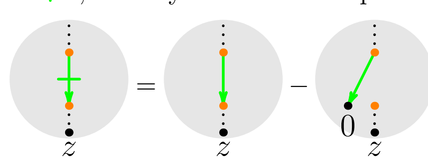
$$\mathcal{W}^{(\epsilon,\|\tau\|-2n)}(\tau) = \sum_{\gamma\in\mathcal{F}(\|\tau\|,n)} C_\gamma(\mathcal{W}^{(\epsilon,\|\tau\|)}(\tau)(z)).$$

Here $\mathcal{F}(\|\tau\|,n)$ denotes the set of graphs consisting of vertexes $1,2,\dots,\|\tau\| \in \mathbb{Z}_+$ and n edges without common vertexes. For a function f on $(\mathbb{R}\times\mathbb{T})^{\|\tau\|}$ and $\gamma \in \mathcal{F}(\|\tau\|,n)$, we define a function $C_\gamma f$ on $(\mathbb{R}\times\mathbb{T})^{\|\tau\|-2n}$ by

$$\begin{aligned}C_\gamma f(u_1,\dots,u_{\|\tau\|-2n}) &= \int_{w_{i_1},\dots,w_{i_n},w_{j_1},\dots,w_{j_n}\in\mathbb{R}\times[0,1)} f(w_1,\dots,w_{\|\tau\|})\\ &\qquad \times \prod_{k=1}^n \delta(w_{i_k}-w_{j_k})dw_{i_k}dw_{j_k}\Big|_{w_{m_1}=u_1,\dots,w_{m_{\|\tau\|-2n}}=u_{\|\tau\|-2n}},\end{aligned}$$

where $\{(i_k,j_k)\}_{k=1}^n$ are all of edges, and $\{m_1 < \cdots < m_{\|\tau\|-2n}\}$ are all of unpaired vertexes. $\square$

For an occurrence of $\leftarrow\!\!+\!\!-$, we may need to decompose a graph as follows.



Then we can write

$$\mathcal{W}_l^{(\epsilon,k)}(\tau)(z) = \sum_i \sigma_{l,i}^k(\tau)\mathcal{W}_{l,i}^{(\epsilon,k)}(\tau)(z)$$

with $\sigma_{l,i}^k(\tau) \in \{1,-1\}$. We write again $\{\mathcal{W}_l^{(\epsilon,k)}\}$ instead of $\{\mathcal{W}_{l,i}^{(\epsilon,k)}\}$, by renumbering its indexes.

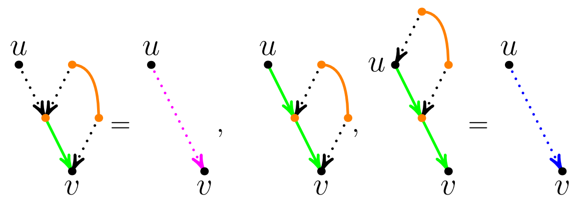
4.5. **Construction of limit graphs.** We introduce two kernels M_ϵ and N_ϵ by

$$M_\epsilon = (K'\pi(\partial_x^\gamma K'_\epsilon * \overleftarrow{\partial_x^\gamma K'_\epsilon})) * \pi\partial_x^\gamma K'_\epsilon, \quad N_\epsilon = (K'\pi(\partial_x^\gamma K'_\epsilon * \overleftarrow{\partial_x^\gamma K'_\epsilon})) * K'. \tag{4.5}$$

Let M and N are functions in Lemma 4.27 and Lemma 4.28. We write these functions as follows.

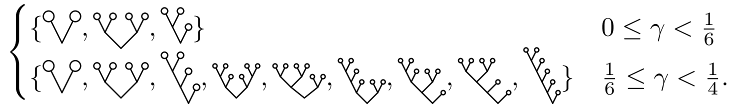
$$z \bullet\!\!\leftarrow\cdots\bullet\, w = M_\epsilon(z-w), \qquad z \bullet\!\!\longleftarrow\!\!\bullet\, w = M(z-w),$$

In graphical notations, M_ϵ and N_ϵ are introduced by



The following lemma is obvious from definitions and using Lemmas 4.3 and 4.6.

Lemma 4.12. *Let*

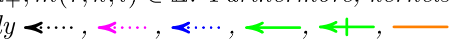
$\mathcal{F}_{\mathrm{re}} =$ 

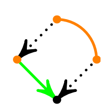
Let $C_\tau^{(\epsilon)}$ $(\tau \in \mathcal{F}_{\mathrm{re}})$ be constants defined in Sections 5 and 7, and $\hat{Z}^{(\epsilon)}$ be the corresponding renormalized model. Then for each $\tau \in \mathcal{F}_-$ and $z \in \mathbb{R}^2$, there exists a family of functions $\{\hat{\mathcal{W}}^{(\epsilon,k)}(\tau)(z) \in H^{\otimes k}\}_{k \in \|\tau\| - 2\mathbb{Z}_+, k \geq 0}$ such that

$$(\hat{\Pi}_0^{(\epsilon)}\tau)(z) = \sum_k I_k(\mathcal{W}^{(\epsilon,k)}(\tau)(z)),$$

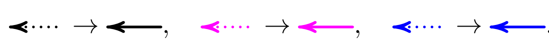
where $\hat{\mathcal{W}}^{(\epsilon,\|\tau\|)}(\tau)(z) = \mathcal{W}^{(\epsilon,\|\tau\|)}(\tau)(z)$ and kernels $\hat{\mathcal{W}}^{(\epsilon,k)}(\tau)$ with $k < \|\tau\|$ can be decomposed into

$$\hat{\mathcal{W}}^{(\epsilon,k)}(\tau)(z) = \sum_{l=1}^{\hat{N}(\tau,k)} \hat{m}(\tau,k,l)\hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)(z)$$

with $\hat{N}(\tau,k) \in \mathbb{Z}_+, \hat{m}(\tau,k,l) \in \mathbb{Z}$. Furthermore, kernels $\hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)$ can be written as graphs using only  *for their edges, without the following form.*



We define the limit graph $\hat{\mathcal{W}}_l^{(k)}(\tau)(z)$ of $\hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)(z)$ by replacing dotted edges as follows.



The following theorem proves Theorems 4.5 and 4.8

Theorem 4.13. *Let $0 \leq \gamma < \frac{1}{4}$. For every $\tau \in \mathcal{F}_-$, k, l, and $\varphi \in \mathcal{B}_0^2$, the function*

$$\int_{\mathbb{R}^2} \varphi(z)\hat{\mathcal{W}}_l^{(k)}(\tau)(z;\cdot)dz$$

belongs to $H^{\otimes k}$. If we define random variables by

$$(\hat{\Pi}_0\tau)(\varphi) = \sum_k \mathcal{I}_k\left(\sum_l \int \varphi(u)\hat{\mathcal{W}}_l^{(k)}(\tau)(u)du\right),$$

then there exist $\iota, \kappa > 0$ such that we have

$$\mathbb{E}|(\hat{\Pi}_0\tau)(\varphi_0^\lambda)| \lesssim \lambda^{2|\tau|_{\mathfrak{s}}+\iota}, \quad \mathbb{E}|(\hat{\Pi}_0\tau - \hat{\Pi}_0^{(\epsilon)}\tau)(\varphi_0^\lambda)| \lesssim \epsilon^{2\kappa}\lambda^{2|\tau|_{\mathfrak{s}}+\iota}$$

uniformly over $0 < \lambda < C$.

proof. For $\tau = \Xi, \mathbf{1}$, by definitions we have

$$(\hat{\Pi}_0 \Xi)(\varphi) = \partial_x^\gamma \xi(\varphi), \quad (\hat{\Pi}_0^{(\epsilon)} \Xi)(\varphi) = \int_{\mathbb{R}^2} \partial_x^\gamma \xi_\epsilon(z) \varphi(z) dz,$$
$$(\hat{\Pi}_0 \mathbf{1})(\varphi) = (\hat{\Pi}_0^{(\epsilon)} \mathbf{1})(\varphi) = \int_{\mathbb{R}^2} \varphi(z) dz.$$

Then the required bounds are trivial for $\mathbf{1}$, and obtained as a result of Lemma 2.2 for Ξ.

For $\tau \neq \Xi, \mathbf{1}$, from Theorem 7.26 of [6] we have

$$\begin{aligned}\mathbb{E}|(\hat{\Pi}_0 \tau)(\varphi_0^\lambda)|^2 &\lesssim \sum_{k,l} \left\| \int_{\mathbb{R}^2} \varphi_0^\lambda(z) \hat{\mathcal{W}}_l^{(k)}(\tau)(z) dz \right\|_{H^{\otimes k}}^2 \\ &= \sum_{k,l} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi_0^\lambda(z) \varphi_0^\lambda(\bar{z}) (\hat{\mathcal{W}}_l^{(k)}(\tau)(z), \hat{\mathcal{W}}_l^{(k)}(\tau)(\bar{z}))_{H^{\otimes k}} dz d\bar{z},\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}|(\hat{\Pi}_0 \tau - \hat{\Pi}_0^{(\epsilon)} \tau)(\varphi_0^\lambda)|^2 \\ &\lesssim \sum_{k,l} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi_0^\lambda(z) \varphi_0^\lambda(\bar{z}) (\delta \hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)(z), \delta \hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)(\bar{z}))_{H^{\otimes k}} dz d\bar{z},\end{aligned}$$

where $\delta \hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau) = \hat{\mathcal{W}}_l^{(k)}(\tau) - \hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)$. We write

$$\mathcal{P}_l^{(k)}(\tau)(z;\bar{z}) = (\hat{\mathcal{W}}_l^{(k)}(\tau)(z), \hat{\mathcal{W}}_l^{(k)}(\tau)(\bar{z}))_{H^{\otimes k}},$$
$$\mathcal{P}_l^{(\epsilon,k)}(\tau)(z;\bar{z}) = (\delta \hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)(z), \delta \hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)(\bar{z}))_{H^{\otimes k}}.$$

In the latter sections, we prove estimates

$$|\mathcal{P}_l^{(k)}(\tau)(z;\bar{z})| \lesssim \begin{cases} \|z - \bar{z}\|_{\mathfrak{s}}^{2|\tau|_{\mathfrak{s}} + \iota} \quad \text{or} \\ \sum_{\zeta_1, \zeta_2 > 0} \|z\|_{\mathfrak{s}}^{\zeta_1} \|\bar{z}\|_{\mathfrak{s}}^{\zeta_2} \|z - \bar{z}\|_{\mathfrak{s}}^{2|\tau|_{\mathfrak{s}} - \zeta_1 - \zeta_2 + \iota} \quad \text{or} \\ \|z\|_{\mathfrak{s}}^{|\tau|_{\mathfrak{s}} + \iota/2} \|\bar{z}\|_{\mathfrak{s}}^{|\tau|_{\mathfrak{s}} + \iota/2} \end{cases}$$

and

$$|\mathcal{P}_l^{(\epsilon,k)}(\tau)(z;\bar{z})| \lesssim \begin{cases} \epsilon^\kappa \|z - \bar{z}\|_{\mathfrak{s}}^{2|\tau|_{\mathfrak{s}} + \iota} \quad \text{or} \\ \epsilon^\kappa \sum_{\zeta_1, \zeta_2 > 0} \|z\|_{\mathfrak{s}}^{\zeta_1} \|\bar{z}\|_{\mathfrak{s}}^{\zeta_2} \|z - \bar{z}\|_{\mathfrak{s}}^{2|\tau|_{\mathfrak{s}} - \zeta_1 - \zeta_2 + \iota} \quad \text{or} \\ \epsilon^\kappa \|z\|_{\mathfrak{s}}^{|\tau|_{\mathfrak{s}} + \iota/2} \|\bar{z}\|_{\mathfrak{s}}^{|\tau|_{\mathfrak{s}} + \iota/2} \end{cases}$$

uniformly over $\epsilon > 0$ and $z, \bar{z} \in B_{\mathfrak{s}}(0, C)$ with some $\kappa, \iota > 0$. Here the sums run over finitely many ζ_1 and ζ_2. Since it turns out that all of negative indexes which appear in the above bounds are greater than -3, they are integrable around $0 \in \mathbb{R}^2$ and we have

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi_0^\lambda(z) \varphi_0^\lambda(\bar{z}) \mathcal{P}_l^{(k)}(\tau)(z;\bar{z}) dz d\bar{z} \lesssim \lambda^{2|\tau|_{\mathfrak{s}} + \iota},$$
$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi_0^\lambda(z) \varphi_0^\lambda(\bar{z}) \mathcal{P}_l^{(\epsilon,k)}(\tau)(z;\bar{z}) dz d\bar{z} \lesssim \epsilon^\kappa \lambda^{2|\tau|_{\mathfrak{s}} + \iota}.$$

$\square$

4.6. **Some calculations of singular kernels.** We give some useful results in order to prove Theorem 4.13.

4.6.1. *Convolutions of kernels.* We frequently use the following estimates.

Lemma 4.14 (Lemma 10.14 of [3]). *Let $\zeta,\eta\in(0,3)$ and $\zeta+\eta>3$.*

(1) Uniformly over $z\in\mathbb{R}^2$, we have

$$\int_{\mathbb{R}^2}\|w\|_{\mathfrak{s}}^{-\zeta}\|z-w\|_{\mathfrak{s}}^{-\eta}dw\lesssim\|z\|_{\mathfrak{s}}^{3-\zeta-\eta}.$$

(2) Uniformly over $z\in\mathbb{R}^2$ and $\epsilon>0$, we have

$$\int_{\mathbb{R}^2}(\epsilon+\|w\|_{\mathfrak{s}})^{-\zeta}\|z-w\|_{\mathfrak{s}}^{-\eta}dw\lesssim(\epsilon+\|z\|_{\mathfrak{s}})^{3-\zeta-\eta}.$$

proof. (1) We divide the domain of the integration as follows.

$$D_1:\|z-w\|_{\mathfrak{s}}\le\frac{1}{2}\|z\|_{\mathfrak{s}},\quad D_2:\|w\|_{\mathfrak{s}}\le\frac{1}{2}\|z\|_{\mathfrak{s}},\quad D_3:\text{otherwise}$$

Since $D_1\subset\{w\,;\|w\|_{\mathfrak{s}}\ge\frac{1}{2}\|z\|_{\mathfrak{s}}\}$ and $D_2\subset\{w\,;\|z-w\|_{\mathfrak{s}}\ge\frac{1}{2}\|z\|_{\mathfrak{s}}\}$, we have

$$\int_{D_1}\|w\|_{\mathfrak{s}}^{-\zeta}\|z-w\|_{\mathfrak{s}}^{-\eta}dw\lesssim\|z\|_{\mathfrak{s}}^{-\zeta}\int_{\|z-w\|_{\mathfrak{s}}\le\frac{1}{2}\|z\|_{\mathfrak{s}}}\|z-w\|_{\mathfrak{s}}^{-\eta}dw\lesssim\|z\|_{\mathfrak{s}}^{3-\zeta-\eta},$$

$$\int_{D_2}\|w\|_{\mathfrak{s}}^{-\zeta}\|z-w\|_{\mathfrak{s}}^{-\eta}dw\lesssim\|z\|_{\mathfrak{s}}^{-\eta}\int_{\|w\|_{\mathfrak{s}}\le\frac{1}{2}\|z\|_{\mathfrak{s}}}\|w\|_{\mathfrak{s}}^{-\zeta}dw\lesssim\|z\|_{\mathfrak{s}}^{3-\zeta-\eta}.$$

For $w\in D_3$, since

$$\|z-w\|_{\mathfrak{s}}=\frac{1}{3}\|z-w\|_{\mathfrak{s}}+\frac{2}{3}\|z-w\|_{\mathfrak{s}}\ge\frac{1}{3}(\|w\|_{\mathfrak{s}}-\|z\|_{\mathfrak{s}})+\frac{1}{3}\|z\|_{\mathfrak{s}}=\frac{1}{3}\|w\|_{\mathfrak{s}},$$

we have

$$\int_{D_3}\|w\|_{\mathfrak{s}}^{-\zeta}\|z-w\|_{\mathfrak{s}}^{-\eta}dw\lesssim\int_{\|w\|_{\mathfrak{s}}\ge\frac{1}{2}\|z\|_{\mathfrak{s}}}\|w\|_{\mathfrak{s}}^{-\zeta-\eta}dw\lesssim\|z\|_{\mathfrak{s}}^{3-\zeta-\eta}.$$

(2) If $\|z\|_{\mathfrak{s}}\ge\epsilon$, from (1) we have

$$\begin{aligned}\int_{\mathbb{R}^2}(\epsilon+\|w\|_{\mathfrak{s}})^{-\zeta}\|z-w\|_{\mathfrak{s}}^{-\eta}dw&\le\int_{\mathbb{R}^2}\|w\|_{\mathfrak{s}}^{-\zeta}\|z-w\|_{\mathfrak{s}}^{-\eta}dw\\&\lesssim\|z\|_{\mathfrak{s}}^{3-\zeta-\eta}\lesssim(\epsilon+\|z\|_{\mathfrak{s}})^{3-\zeta-\eta}.\end{aligned}$$

If $\|z\|_{\mathfrak{s}}\le\epsilon$, we divide the domain of the integration into $\|w\|_{\mathfrak{s}}\le2\epsilon$ and $\|w\|_{\mathfrak{s}}>2\epsilon$. For $\|w\|_{\mathfrak{s}}\le2\epsilon$, since $\|z-w\|_{\mathfrak{s}}\le3\epsilon$ we have

$$\begin{aligned}\int_{\|w\|_{\mathfrak{s}}\le2\epsilon}(\epsilon+\|w\|_{\mathfrak{s}})^{-\zeta}\|z-w\|_{\mathfrak{s}}^{-\eta}dw&\le\epsilon^{-\zeta}\int_{\|z-w\|_{\mathfrak{s}}\le3\epsilon}\|z-w\|_{\mathfrak{s}}^{-\eta}dw\\&\lesssim\epsilon^{3-\zeta-\eta}\lesssim(\epsilon+\|z\|_{\mathfrak{s}})^{3-\zeta-\eta}.\end{aligned}$$

For $\|w\|_{\mathfrak{s}}>2\epsilon$, since $\|z-w\|_{\mathfrak{s}}\ge\|w\|_{\mathfrak{s}}-\epsilon\ge\frac{1}{2}\|w\|_{\mathfrak{s}}$ we have

$$\begin{aligned}\int_{\|w\|_{\mathfrak{s}}>2\epsilon}(\epsilon+\|w\|_{\mathfrak{s}})^{-\zeta}\|z-w\|_{\mathfrak{s}}^{-\eta}dw&\lesssim\int_{\|w\|_{\mathfrak{s}}>2\epsilon}\|w\|_{\mathfrak{s}}^{-\zeta-\eta}dw\\&\lesssim\epsilon^{3-\zeta-\eta}\lesssim(\epsilon+\|z\|_{\mathfrak{s}})^{3-\zeta-\eta}.\end{aligned}$$

□

Lemma 4.15. *Let $\zeta,\eta\in(0,3)$ and $C>0$. If $\zeta+\eta<3$, uniformly over $z\in\mathbb{R}^2$ we have*

$$\int_{\|w\|_{\mathfrak{s}},\|z-w\|_{\mathfrak{s}}\le C}\|w\|_{\mathfrak{s}}^{-\zeta}\|z-w\|_{\mathfrak{s}}^{-\eta}dw\lesssim1.$$

proof. From Young's inequality, we have

$$\int_{\|w\|_{\mathfrak{s}},\|z-w\|_{\mathfrak{s}}\le C}\|w\|_{\mathfrak{s}}^{-\zeta}\|z-w\|_{\mathfrak{s}}^{-\eta}dw \lesssim \int_{\|w\|_{\mathfrak{s}}\le C}\|w\|_{\mathfrak{s}}^{-\zeta-\eta}dw+\int_{\|z-w\|_{\mathfrak{s}}\le C}\|z-w\|_{\mathfrak{s}}^{-\zeta-\eta}dw\lesssim C^{3-\zeta-\eta}.$$

□

4.6.2. *Estimates of K.* Let K be a function defined in Subsection 3.4. The following estimates is obtained from Lemmas 5.5 and 10.17 of [3].

Lemma 4.16. *For every $k\in\mathbb{Z}_+^2$, we have $|\partial^k K(z)|\lesssim\|z\|_{\mathfrak{s}}^{-1-|k|_{\mathfrak{s}}}$. Furthermore, let $\rho\in C_0^\infty(\mathbb{R}^2)$ be a function such that $\int\rho=1$ and set $\rho_\epsilon=\rho_0^\epsilon$ and $K_\epsilon=K*\rho_\epsilon$. Then we have*

$$|\partial^k K_\epsilon(z)|\lesssim(\epsilon+\|z\|_{\mathfrak{s}})^{-1-|k|_{\mathfrak{s}}},$$

and for every $\theta\in(0,1]$ we have

$$|\partial^k K(z)-\partial^k K_\epsilon(z)|\lesssim\epsilon^\theta\|z\|_{\mathfrak{s}}^{-1-|k|_{\mathfrak{s}}-\theta}.$$

4.6.3. *Estimates of $\partial_x^\gamma K'$.* For a function A on $\mathbb{R}^2\setminus\{0\}$, we write

$$\partial_x^\gamma A(t,x):=\begin{cases}(-\Delta)^{\frac{\gamma}{2}}(A(t,\cdot))(x) & (t\neq0)\\ 0 & (t=0).\end{cases}$$

Lemma 4.17. *Let $0<\gamma<1$. For every $k\in\mathbb{Z}_+^2$, we have $|\partial_x^\gamma\partial^k K(z)|\lesssim\|z\|_{\mathfrak{s}}^{-1-|k|_{\mathfrak{s}}-\gamma}$.*

proof. First we consider $\partial_x^\gamma\partial^k G$. Since $\partial_t G(t,\cdot)\equiv\partial_x^2 G(t,\cdot)$ for every $t>0$, we have $\partial_x^\gamma\partial^k G=\partial_x^\gamma\partial_x^{|k|_{\mathfrak{s}}}G$. Set $\varphi(x)=\frac{1}{\sqrt{4\pi}}e^{-\frac{x^2}{4}}$, so that we have

$$\partial_x^\gamma\partial_x^{|k|_{\mathfrak{s}}}G(t,x)=\mathbf{1}_{t>0}\frac{1}{\sqrt{t}^{1+|k|_{\mathfrak{s}}+\gamma}}(-\Delta)^{\frac{\gamma}{2}}\phi^{(|k|_{\mathfrak{s}})}\left(\frac{x}{\sqrt{t}}\right).$$

Since $(-\Delta)^{\frac{\gamma}{2}}\varphi^{(|k|_{\mathfrak{s}})}$ is bounded, we have $\partial_x^\gamma\partial_x^{|k|_{\mathfrak{s}}}G(t,x)\lesssim\sqrt{|t|}^{-1-|k|_{\mathfrak{s}}-\gamma}$. Since $|(-\Delta)^{\frac{\gamma}{2}}\varphi^{(|k|_{\mathfrak{s}})}(x)|\lesssim|x|^{-1-|k|_{\mathfrak{s}}-\gamma}$ from Lemma 2.5, we have

$$|\partial_x^\gamma\partial_x^{|k|_{\mathfrak{s}}}G(t,x)|\lesssim\frac{1}{\sqrt{|t|}^{1+|k|_{\mathfrak{s}}+\gamma}}\left(\frac{\sqrt{|t|}}{|x|}\right)^{1+|k|_{\mathfrak{s}}+\gamma}\lesssim|x|^{-1-|k|_{\mathfrak{s}}-\gamma}.$$

These estimates imply $|\partial_x^\gamma G'(z)|\lesssim\|z\|_{\mathfrak{s}}^{-1-|k|_{\mathfrak{s}}-\gamma}$.

Next we consider $\partial_x^\gamma\partial^k R$. Since $\partial^k R$ and $\partial_x\partial^k R$ are bounded, $\partial_x^\gamma\partial^k R$ is bounded from Lemma 2.3. We verify that for every $n\ge1$

$$\sup_x|x|^n|\partial^k R(t,x)|\lesssim1$$

locally uniformly over t. If $|x|\le C$, this is obvious because $\partial^k R$ is bounded. If $|x|\ge C$, since $\partial^k R(t,x)=\partial^k G(t,x)=\partial_x^{|k|_{\mathfrak{s}}}G(t,x)$ we have

$$\sup_{|x|\ge C}|x|^n|\partial^k R(t,x)|\lesssim\sup_x\left|\mathbf{1}_{t>0}\frac{|x|^{n+|k|_{\mathfrak{s}}}}{\sqrt{t}^{1+|k|_{\mathfrak{s}}}}e^{-\frac{x^2}{4t}}\right|$$

$$= \sup_x \left| \mathbf{1}_{t>0} \sqrt{t}^{n-1} \left(\frac{|x|}{\sqrt{t}} \right)^{n+|k|_{\mathfrak{s}}} e^{-\frac{1}{4}\left(\frac{|x|}{\sqrt{t}}\right)^2} \right| \lesssim \mathbf{1}_{t>0} \sqrt{t}^{n-1}.$$

Therefore, from Lemma 2.5 we have $|\partial_x^\gamma \partial^k R(t,x)| \lesssim 1 \wedge |x|^{-1-|k|_{\mathfrak{s}}-\gamma}$ locally uniformly over t.

From these estimates and the fact that $\partial_x^\gamma \partial^k K$ is supported in $\sqrt{|t|} \le C$, we have the required bound. $\square$

Remark 4.18. *We have $|\partial_x^\gamma \partial^k R| \lesssim \|z\|_{\mathfrak{s}}^{-1-|k|_{\mathfrak{s}}-\gamma}$ from the bounds of $\partial_x^\gamma \partial^k G$ and $\partial_x^\gamma \partial^k K$.*

Lemma 4.19. *Let $0 < \gamma < 1$.*

(1) We have $(\partial_x^\gamma K')' = \partial_x^\gamma K''$ and $|(\partial_x^\gamma K')'(z)| \lesssim \|z\|_{\mathfrak{s}}^{-3-\gamma}$.

(2) $\partial_x^\gamma K'$ is smooth except at $t=0$ and we have

$$\partial_t(\partial_x^\gamma K')(t,x) = \partial_x^\gamma(\partial_t K')(t,x), \quad |\partial_t(\partial_x^\gamma K')(z)| \lesssim \|z\|_{\mathfrak{s}}^{-4-\gamma} \quad (t \ne 0).$$

proof. (1) is obtained from Lemma 4.17.

(2) Since the map $t \mapsto K'(t,\cdot) \in \mathcal{S}(\mathbb{R})$ is continuous except at $t=0$, $\partial_t \partial_x^\gamma K' = \partial_x^\gamma \partial_t K'$ holds on $t \ne 0$. The estimate is obtained from Lemma 4.17. $\square$

4.6.4. *Estimates of L.* Since $\partial_x^\gamma K'(t,x)$ decreases sufficiently fast as $|x| \to \infty$, we can define $\pi \partial_x^\gamma K'(t,x) = \sum_{n\in\mathbb{Z}} \partial_x^\gamma K'(t,x+n)$. We set

$$L(z) = \int_{w \in \mathbb{R}\times[0,1)} \pi \partial_x^\gamma K'(z-w) \pi \partial_x^\gamma K'(-w) dw = \pi(\partial_x^\gamma K' * \overleftarrow{\partial_x^\gamma K'})(z).$$

Lemma 4.20. *Let $0 < \gamma < \frac{1}{2}$. There exists a function L_0 such that*

$$L = \pi L_0, \quad |L_0(z)| \lesssim \|z\|_{\mathfrak{s}}^{-1-2\gamma}.$$

proof. Since $L = \pi(\partial_x^\gamma K' * \overleftarrow{\partial_x^\gamma K'}) = \pi((\partial_x^{2\gamma} K') * \overleftarrow{K'})$, set $L_0 = (\partial_x^{2\gamma} K') * \overleftarrow{K'}$. From the estimates of $\partial_x^{2\gamma} K'$ and K', we have the required bound from Lemma 4.14. $\square$

Let $\rho \in C_0^\infty(\mathbb{R}^2)$ be a function such that $\int \rho = 1$. Set $K'_\epsilon = K' * \rho_\epsilon$ and define

$$L_\epsilon^{(1)} = \int_{w \in \mathbb{R}\times[0,1)} \pi \partial_x^\gamma K'_\epsilon(z-w) \pi \partial_x^\gamma K'(-w) dw = \pi(\partial_x^\gamma K'_\epsilon * \overleftarrow{\partial_x^\gamma K'})(z),$$

$$L_\epsilon^{(2)} = \int_{w \in \mathbb{R}\times[0,1)} \pi \partial_x^\gamma K'_\epsilon(z-w) \pi \partial_x^\gamma K'_\epsilon(-w) dw = \pi(\partial_x^\gamma K'_\epsilon * \overleftarrow{\partial_x^\gamma K'_\epsilon})(z).$$

Lemma 4.21. *Let $0 < \gamma < \frac{1}{2}$. There exists a function $L_{\epsilon,0}^{(i)}$ $(i=1,2)$ such that*

$$L_\epsilon^{(i)} = \pi L_{\epsilon,0}^{(i)}, \quad |L_{\epsilon,0}^{(i)}(z)| \lesssim (\epsilon + \|z\|_{\mathfrak{s}})^{-1-2\gamma}.$$

Furthermore, for every $\theta \in (0,1]$ we have

$$|L_0(z) - L_{\epsilon,0}^{(1)}(z)| \lesssim \epsilon^\theta \|z\|_{\mathfrak{s}}^{-1-2\gamma-\theta}, \quad |L_{\epsilon,0}^{(1)}(z) - L_{\epsilon,0}^{(2)}(z)| \lesssim \epsilon^\theta \|z\|_{\mathfrak{s}}^{-1-2\gamma-\theta}.$$

proof. Since $L_\epsilon^{(1)} = \pi(\partial_x^{2\gamma} K' * \overleftarrow{K'_\epsilon})$ and $L_\epsilon^{(2)} = \pi(\partial_x^{2\gamma} K' * (\overleftarrow{K'_\epsilon} * \rho_\epsilon))$, set

$$L_{\epsilon,0}^{(1)} = \partial_x^{2\gamma} K' * \overleftarrow{K'_\epsilon}, \quad L_{\epsilon,0}^{(2)} = \partial_x^{2\gamma} K' * (\overleftarrow{K'_\epsilon} * \rho_\epsilon).$$

From the estimates of $\overleftarrow{K'_\epsilon}$ and $\overleftarrow{K'_\epsilon} * \rho_\epsilon$, we obtain the bounds of $L_{\epsilon,0}^{(i)}$ $(i=1,2)$. Since

$$L_0 - L_{\epsilon,0}^{(1)} = \partial_x^{2\gamma} K' * (\overleftarrow{K'} - \overleftarrow{K'_\epsilon}), \quad L_{\epsilon,0}^{(1)} - L_{\epsilon,0}^{(2)} = \partial_x^{2\gamma} K' * (\overleftarrow{K'_\epsilon} - \overleftarrow{K'_\epsilon} * \rho_\epsilon),$$

we obtain the convergence results from the estimates of $\overleftarrow{K'}-\overleftarrow{K'_\epsilon}$ and $\overleftarrow{K'_\epsilon}-\overleftarrow{K'_\epsilon}*\rho_\epsilon$. □

4.6.5. *Estimates of M and N.* In this subsection, we assume that K' is supported in $B_{\mathfrak{s}}(0,\frac{1}{2})$, and ρ is even in x and supported in $B_{\mathfrak{s}}(0,1)$. Since K' is odd and $L_\epsilon^{(2)}$ is even in x, for every $t\in\mathbb{R}$ we have

$$\int_{\mathbb{R}}(K'L_\epsilon^{(2)})*\rho_\epsilon(t,x)dx=0. \tag{4.6}$$

Let M_ϵ and N_ϵ be functions defined in (4.5), i.e.

$$M_\epsilon=(K'L_\epsilon^{(2)})*\pi\partial_x^\gamma K'*\rho_\epsilon=\pi((K'L_\epsilon^{(2)})*\rho_\epsilon*\partial_x^\gamma K'),$$
$$N_\epsilon=(K'L_\epsilon^{(2)})*K'*\rho_\epsilon=(K'L_\epsilon^{(2)})*\rho_\epsilon*K'.$$

Lemma 4.22. *For every $x\neq 0$, there exists a limit*

$$\partial_x^\gamma K'(0+,x):=\lim_{t\downarrow 0}\partial_x^\gamma K'(t,x).$$

proof. For a fixed $(t,x)\in(0,\infty)\times\mathbb{R}$, set $c_n=\partial_x^\gamma K'(2^{-n}t,x)$ $(n\in\mathbb{Z}_+)$. From Lemma 4.19 (2), we have

$$|c_n-c_{n+1}|\lesssim|2^{-n}t-2^{-n-1}t|\;|x|^{-4-\gamma}=2^{-n-1}|t|\;|x|^{-4-\gamma}.$$

Hence there exists $c=\lim_{n\to\infty}c_n$. This limit is independent of t. □

Lemma 4.23. *Define $\widetilde{\partial_x^\gamma K'}:\mathbb{R}^2\setminus\{0\}\to\mathbb{R}$ by*

$$\widetilde{\partial_x^\gamma K'}(t,x)=\begin{cases}\partial_x^\gamma K'(t,x) & (t>0)\\ \partial_x^\gamma K'(0+,x) & (t=0,x\neq 0)\\ \partial_x^\gamma K'(-t,x) & (t<0).\end{cases}$$

Then we have

$$|\widetilde{\partial_x^\gamma K'}(z)-\widetilde{\partial_x^\gamma K'}(w)|\lesssim\|z-w\|_{\mathfrak{s}}\|z\|_{\mathfrak{s}}^{-3-\gamma}$$

uniformly over z,w with $\|z-w\|_{\mathfrak{s}}\le\frac{\|z\|_{\mathfrak{s}}}{2}$.

proof. Let $z=(t,x),w=(s,y)\in\mathbb{R}^2$. Without loss of generality, we can assume that $t\ge 0$. If $t,s>0$, from Lemma 4.19 we have

$$|\widetilde{\partial_x^\gamma K'}(t,x)-\widetilde{\partial_x^\gamma K'}(s,y)|\lesssim|t-s|\;\|z\|_{\mathfrak{s}}^{-4-\gamma}+|x-y|\;\|z\|_{\mathfrak{s}}^{-3-\gamma}\lesssim\|z-w\|_{\mathfrak{s}}\|z\|_{\mathfrak{s}}^{-3-\gamma}.$$

If $t=0$ or $s=0$, we obtain the required bound by taking limit $t,s\to 0+$. When $t\ge 0>s$, since $\bar{w}=(-s,y)$ also satisfies $\|z-\bar{w}\|_{\mathfrak{s}}\le\frac{\|z\|_{\mathfrak{s}}}{2}$ we have

$$|\widetilde{\partial_x^\gamma K'}(t,x)-\widetilde{\partial_x^\gamma K'}(s,y)|=|\widetilde{\partial_x^\gamma K'}(t,x)-\widetilde{\partial_x^\gamma K'}(-s,y)|\lesssim\|z-w\|_{\mathfrak{s}}\|z\|_{\mathfrak{s}}^{-3-\gamma}.$$

□

We need the Lipschitz continuities of $L_\epsilon^{(2)}$ and K'.

Lemma 4.24. *For every $z,w\in B_{\mathfrak{s}}(0,\frac{1}{2})$ with $\|z-w\|_{\mathfrak{s}}\le\frac{\|z\|_{\mathfrak{s}}}{2}$, we have*

$$|L_\epsilon^{(2)}(z)-L_\epsilon^{(2)}(w)|\lesssim\|z-w\|_{\mathfrak{s}}\|z\|_{\mathfrak{s}}^{-2-2\gamma}.$$

proof. Let $z=(t,x), w=(s,y)\in\mathbb{R}^2$. Since $K=G-R$, we have

$$
\begin{aligned}
L_0(z) &= \partial_x^{2\gamma}(K' * \overleftarrow{K'})(z)\\
&= \partial_x^{2\gamma}(G' * \overleftarrow{G'})(z) - K' * \partial_x^{2\gamma}\overleftarrow{R'}(z) - \partial_x^{2\gamma}R' * \overleftarrow{K'}(z) - (\partial_x^{2\gamma}R') * \overleftarrow{R'}(z).
\end{aligned}
$$

Since the last three terms are convolutions of integrable kernels and integrable smooth functions, they are smooth and have bounded derivatives.

We consider the first term. From the semigroup property, $G(t,\cdot) *_x G(s,\cdot) = G(t+s,\cdot)$ for every $t,s>0$. For every $t>0$, we have

$$
\begin{aligned}
G' * \overleftarrow{G'}(t,x) &= \iint G'(t-s,x-y)G'(-s,-y)dsdy\\
&= \int_{s<0}\left(\int G'(t-s,x-y)G'(-s,y)dy\right)ds\\
&= \int_{s<0} G''(t-2s,x)ds = \int_{s<0}\partial_t G(t-2s,x)ds = -\frac{1}{2}G(t,x).
\end{aligned}
$$

For every $t<0$, we have

$$
\begin{aligned}
G' * \overleftarrow{G'}(t,x) &= \int_{s<t}\left(\int G'(t-s,x-y)G'(-s,y)dy\right)ds\\
&= \int_{s<t} G''(t-2s,x)ds = \int_{s<t}\partial_t G(t-2s,x)ds = -\frac{1}{2}G(-t,x).
\end{aligned}
$$

Hence we have $G' * \overleftarrow{G'}(t,x) = -\frac{1}{2}G(|t|,x)$ for every $t\neq 0$ and x. Similarly to Lemma 4.23, we have

$$|L_0(z)-L_0(w)| \lesssim \|z-w\|_{\mathfrak{s}}\|z\|_{\mathfrak{s}}^{-2-2\gamma}.$$

We have the same estimate for $L_{\epsilon,0}^{(\epsilon)}$, similarly to Lemma 10.17 of [3], so that for every $n\in\mathbb{Z}$ we have

$$|L_{\epsilon,0}^{(2)}(t,x+n) - L_{\epsilon,0}^{(2)}(s,y+n)| \lesssim \|z-w\|_{\mathfrak{s}}\|(t,x+n)\|_{\mathfrak{s}}^{-2-2\gamma}.$$

Summing these estimates for every $n\in\mathbb{Z}$, we have

$$|L_\epsilon^{(2)}(z) - L_\epsilon^{(2)}(w)| \lesssim \|z-w\|_{\mathfrak{s}}\|z\|_{\mathfrak{s}}^{-2-2\gamma}.$$

□

Lemma 4.25 (Lemma 10.18 of [3]). *For every $\theta\in(0,1]$, we have*

$$|K'(z)-K'(w)| \lesssim \|z-w\|_{\mathfrak{s}}^{\theta}(\|z\|_{\mathfrak{s}}^{-2-\theta} + \|w\|_{\mathfrak{s}}^{-2-\theta}).$$

From the above two lemmas, we have the following estimates.

Lemma 4.26. *Let $(K'L_\epsilon^{(2)})_\epsilon = (K'L_\epsilon^{(2)}) * \rho_\epsilon$. Then we have*

$$|(K'L_\epsilon^{(2)})_\epsilon(z)| \lesssim (\epsilon + \|z\|_{\mathfrak{s}})^{-3-2\gamma}.$$

Furthermore, for every $\theta\in(0,1]$ we have

$$|(K'L_\epsilon^{(2)})_\epsilon(z) - K'(z)L(z)| \lesssim \epsilon^\theta\|z\|_{\mathfrak{s}}^{-3-2\gamma-\theta}.$$

proof. We note that $|K'L_\epsilon^{(2)}(z)| \lesssim \|z\|_{\mathfrak{s}}^{-3-2\gamma}$, since K' is supported in $B_{\mathfrak{s}}(0,\frac{1}{2})$.

First we prove the bound of $(K'L_\epsilon^{(\epsilon)})_\epsilon$. If $\|z\|_{\mathfrak{s}} \geq 2\epsilon$, since $\|w\|_{\mathfrak{s}} \leq \epsilon \Rightarrow \|z-w\|_{\mathfrak{s}} \geq \|z\|_{\mathfrak{s}}/2$ we have

$$|(K'L_\epsilon^{(2)})_\epsilon(z)| \leq \int_{\mathbb{R}^2} |K'L_\epsilon^{(2)}(z-w)|\ |\rho_\epsilon(w)|dw \lesssim \|z\|_{\mathfrak{s}}^{-3-2\gamma}.$$

If $\|z\|_{\mathfrak{s}} \leq 2\epsilon$, since $\|\rho_\epsilon\|_{L^\infty} \lesssim \epsilon^{-3}$ and $\|L_\epsilon^{(2)}\|_{L^\infty} \lesssim \epsilon^{-1-2\gamma}$ we have

$$\begin{aligned}|(K'L_\epsilon^{(2)})_\epsilon(z)| &\leq \int_{\mathbb{R}^2} |K'(z-w)|\ |L_\epsilon^{(2)}(z-w)|\ |\rho_\epsilon(w)|dw \\ &\lesssim \epsilon^{-4-2\gamma} \int_{\|z-w\|_{\mathfrak{s}}\leq 3\epsilon} \|z-w\|_{\mathfrak{s}}^{-2} dw \lesssim \epsilon^{-3-2\gamma}.\end{aligned}$$

Next we prove the convergence result. Since $|K'L_\epsilon^{(2)}(z) - K'L(z)| \lesssim \epsilon^\theta \|z\|_{\mathfrak{s}}^{-3-2\gamma-\theta}$, it is sufficient to prove

$$|(K'L_\epsilon^{(2)})_\epsilon(z) - K'(z)L_\epsilon^{(2)}(z)| \lesssim \epsilon^\theta \|z\|_{\mathfrak{s}}^{-3-2\gamma-\theta}.$$

If $\|z\|_{\mathfrak{s}} \leq 2\epsilon$, we have

$$|(K'L_\epsilon^{(2)})_\epsilon(z) - K'(z)L_\epsilon^{(2)}(z)| \lesssim \|z\|_{\mathfrak{s}}^{-3-2\gamma} \lesssim \epsilon^\theta \|z\|_{\mathfrak{s}}^{-3-2\gamma-\theta}.$$

If $\|z\|_{\mathfrak{s}} \geq 2\epsilon$, we have

$$\begin{aligned}&|(K'L_\epsilon^{(2)})_\epsilon(z) - K'(z)L_\epsilon^{(2)}(z)| \\ &\leq \int |K'(z-w)L_\epsilon^{(2)}(z-w) - K'(z)L_\epsilon^{(2)}(z)|\ |\rho_\epsilon(w)|dw \\ &\lesssim \int \|w\|_{\mathfrak{s}} \|z-w\|_{\mathfrak{s}}^{-4-2\gamma} |\rho_\epsilon(w)|dw \lesssim \epsilon \|z\|_{\mathfrak{s}}^{-4-2\gamma} \lesssim \epsilon^\theta \|z\|_{\mathfrak{s}}^{-3-2\gamma-\theta}.\end{aligned}$$

□

We prove the convergence results of M_ϵ and N_ϵ.

Lemma 4.27. *There exist functions M, M_0 and $M_{\epsilon,0}$ such that $M = \pi M_0$ and $M_\epsilon = \pi M_{\epsilon,0}$, and we have*

$$|M_0(z)| \lesssim \|z\|_{\mathfrak{s}}^{-2-3\gamma}, \quad |M_0(z) - M_{\epsilon,0}(z)| \lesssim \epsilon^\theta \|z\|_{\mathfrak{s}}^{-2-3\gamma-\theta} \quad (\theta \in (0,1]).$$

proof. From (4.6) and Lemma 4.23, we write

$$\begin{aligned}M_{\epsilon,0}(t,x) &= \iint (K'L_\epsilon^{(2)})_\epsilon(t-s,x-y)\partial_x^\gamma K'(s,y)dsdy \\ &= \iint \mathbf{1}_{s>0}(K'L_\epsilon^{(2)})_\epsilon(t-s,x-y)\widetilde{\partial_x^\gamma K'}(s,y)dsdy \\ &= \iint \mathbf{1}_{s>0}(K'L_\epsilon^{(2)})_\epsilon(t-s,x-y)(\widetilde{\partial_x^\gamma K'}(s,y) - \widetilde{\partial_x^\gamma K'}(t,x))dsdy.\end{aligned}$$

We verify that the integral

$$M_0(t,x) = \iint \mathbf{1}_{s>0}(K'L)(t-s,x-y)(\widetilde{\partial_x^\gamma K'}(s,y) - \widetilde{\partial_x^\gamma K'}(t,x))dsdy$$

is well-defined. We divide the domain of the integration as follows.

$$D_1 : \|z-w\|_{\mathfrak{s}} \leq \frac{1}{2}\|z\|_{\mathfrak{s}}, \quad D_2 : \|w\|_{\mathfrak{s}} \leq \frac{1}{2}\|z\|_{\mathfrak{s}}, \quad D_3 : \text{otherwise}.$$

Since $D_1 \subset \{w\,;\|w\|_{\mathfrak{s}} \geq \frac{1}{2}\|z\|_{\mathfrak{s}}\}$ and $D_2 \subset \{w\,;\|z-w\|_{\mathfrak{s}} \geq \frac{1}{2}\|z\|_{\mathfrak{s}}\}$, we have

$$\begin{aligned}
&\int_{D_1} \left|(K'L)(z-w)(\widetilde{\partial_x^\gamma K'}(w) - \widetilde{\partial_x^\gamma K'}(z))\right| dw \\
&\lesssim \int_{\|z-w\|_{\mathfrak{s}} \leq \frac{1}{2}\|z\|_{\mathfrak{s}}} \|z-w\|_{\mathfrak{s}}^{-3-2\gamma}\|z-w\|_{\mathfrak{s}}\|z\|_{\mathfrak{s}}^{-2-\gamma} dw \lesssim \|z\|_{\mathfrak{s}}^{-2-3\gamma},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{D_2} \left|(K'L)(z-w)(\widetilde{\partial_x^\gamma K'}(w) - \widetilde{\partial_x^\gamma K'}(z))\right| dw \\
&\lesssim \int_{D_2} \|z-w\|_{\mathfrak{s}}^{-3-2\gamma}(\|w\|_{\mathfrak{s}}^{-2-\gamma} + \|z\|_{\mathfrak{s}}^{-2-\gamma}) dw \\
&\lesssim \int_{\|w\|_{\mathfrak{s}} \leq \frac{1}{2}\|z\|_{\mathfrak{s}}} \|z\|_{\mathfrak{s}}^{-3-2\gamma}\|w\|_{\mathfrak{s}}^{-2-\gamma} dw + \int_{\|z-w\|_{\mathfrak{s}} \geq \frac{1}{2}\|z\|_{\mathfrak{s}}} \|z-w\|_{\mathfrak{s}}^{-3-2\gamma}\|z\|_{\mathfrak{s}}^{-2-\gamma} dw \\
&\lesssim \|z\|_{\mathfrak{s}}^{-2-3\gamma}.
\end{aligned}$$

Since $D_3 \subset \{w\,;\|z-w\|_{\mathfrak{s}} \geq \frac{1}{3}\|w\|_{\mathfrak{s}}\}$ we have

$$\begin{aligned}
&\int_{D_3} \left|(K'L)(z-w)(\widetilde{\partial_x^\gamma K'}(w) - \widetilde{\partial_x^\gamma K'}(z))\right| dw \\
&\lesssim \int_{\|w\|_{\mathfrak{s}} \geq \frac{1}{2}\|z\|_{\mathfrak{s}}} \|w\|_{\mathfrak{s}}^{-3-2\gamma}(\|w\|_{\mathfrak{s}}^{-2-\gamma} + \|z\|_{\mathfrak{s}}^{-2-\gamma}) dw \lesssim \|z\|_{\mathfrak{s}}^{-2-3\gamma}.
\end{aligned}$$

The estimate of $M_0 - M_{\epsilon,0}$ is obtained from the estimate of $(K'L_\epsilon^{(2)})_\epsilon - K'L$ instead of $K'L$. $\square$

Lemma 4.28. *There exists a function N such that we have*

$$|N(z)| \lesssim \|z\|_{\mathfrak{s}}^{-2-2\gamma}, \quad |N(z) - N_\epsilon(z)| \lesssim \epsilon^\theta \|z\|_{\mathfrak{s}}^{-2-2\gamma-\theta} \quad (\theta \in (0,1]).$$

proof. Similarly to Lemma 4.27, We have

$$\begin{aligned}
N_\epsilon(t,x) &= \iint (K'L_\epsilon^{(2)})_\epsilon(t-s,x-y)K'(s,y)dsdy \\
&= \iint (K'L_\epsilon^{(2)})_\epsilon(t-s,x-y)K'(s,y)dsdy \\
&= \iint (K'L_\epsilon^{(2)})_\epsilon(t-s,x-y)(K'(s,y) - K'(t,x))dsdy.
\end{aligned}$$

Hence we obtain the required results by defining

$$N(t,x) = \iint (K'L)(t-s,x-y)(K'(s,y) - K'(t,x))dsdy.$$

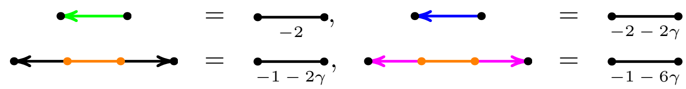
$\square$

4.7. **Contractions of $\mathcal{P}_l^{(k)}(\tau)(z;\bar{z})$.** We introduce labelled graphs to consider orders of divergences of kernels.

Let $\alpha \in \mathbb{R}$. For a compactly supported function $A : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that $\sup_z \|z\|_{\mathfrak{s}}^{-\alpha}|A(z)| < \infty$, we write

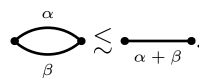
$$A(z-\bar{z}) = z \underset{\alpha}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet} \bar{z}\,.$$

From calculations in Subsection 4.6, each edge has the following label.

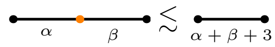


Furthermore we can contract labelled graphs as follows.

(1) (Product) For every $\alpha, \beta \in \mathbb{R}$,

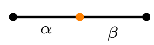


(2) (Convolution) Let $\alpha, \beta \in (-3, 0)$. If $\alpha + \beta < -3$,



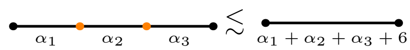
(Lemma 4.14).

If $\alpha + \beta > -3$,

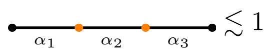


$\lesssim 1$ (Lemma 4.15).

Remark 4.29. *We can also use contractions (2) for more than three kernels. If $\alpha_1, \alpha_2, \alpha_3 \in (-3, 0)$ and $\alpha_1 + \alpha_2 + \alpha_3 < -6$, we can derive*



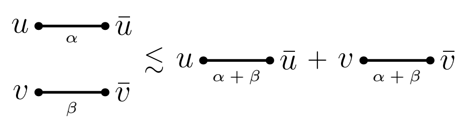
because any of $\alpha_i + \alpha_j$ ($i \neq j$) is smaller than -3, and convolutions are commutative. If $\alpha_1 + \alpha_2 + \alpha_3 > -6$ we obtain



because if $\alpha_1 + \alpha_2 > -3$, a convolution of 1 and a kernel of order α_3 is bounded.

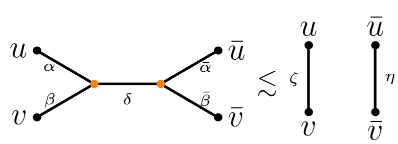
However these contractions are not sufficient. We need the following lemmas in order to contract complicated graphs.

Lemma 4.30. *For every $\alpha, \beta \leq 0$,*



proof. This is a consequence of Young's inequality. □

Lemma 4.31. *Let $\alpha, \beta, \delta, \bar{\alpha}, \bar{\beta} \in (-3, 0)$, $\zeta \in [\alpha \vee \beta, 0)$ and $\eta \in [\bar{\alpha} \vee \bar{\beta}, 0)$. If $\zeta < \alpha + \beta + 3$, $\eta < \bar{\alpha} + \bar{\beta} + 3$ and $\zeta + \eta < \alpha + \beta + \delta + \bar{\alpha} + \bar{\beta} + 6$, then*



proof. Let $a, b, c \geq 0$ such that $a \leq b + c$ and $\theta > 0$. Then we have

$$b^{-\theta} c^{-\theta} \leq 2^{\theta} a^{-\theta} (b^{-\theta} + c^{-\theta}),$$

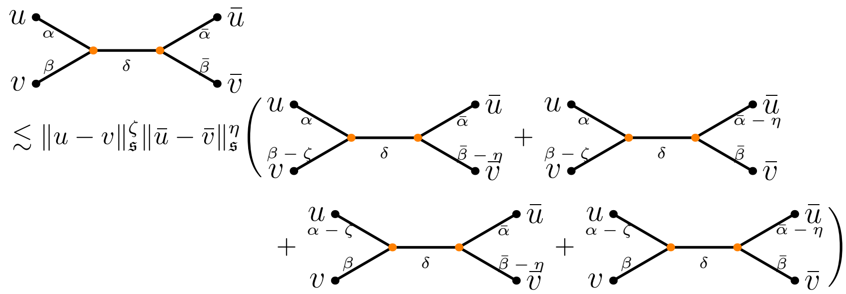
since $a^{\theta} \leq 2^{\theta}(b^{\theta} + c^{\theta})$. By putting $a = \|u - v\|_{\mathfrak{s}}, b = \|u - \bullet\|_{\mathfrak{s}}, c = \|v - \bullet\|_{\mathfrak{s}}$ and $\theta = -\zeta$, we have

(4.7)
$$\begin{aligned}\|u - \bullet\|_{\mathfrak{s}}^{\alpha} \|v - \bullet\|_{\mathfrak{s}}^{\beta} &= \|u - \bullet\|_{\mathfrak{s}}^{\alpha-\zeta} \|v - \bullet\|_{\mathfrak{s}}^{\beta-\zeta} \times \|u - \bullet\|_{\mathfrak{s}}^{\zeta} \|v - \bullet\|_{\mathfrak{s}}^{\zeta} \\ &\lesssim \|u - \bullet\|_{\mathfrak{s}}^{\alpha-\zeta} \|v - \bullet\|_{\mathfrak{s}}^{\beta-\zeta} \times \|u - v\|_{\mathfrak{s}}^{\zeta} (\|u - \bullet\|_{\mathfrak{s}}^{\zeta} + \|v - \bullet\|_{\mathfrak{s}}^{\zeta})\end{aligned}$$

$$= \|u - v\|_{\mathfrak{s}}^{\zeta}(\|u - \bullet\,\|_{\mathfrak{s}}^{\alpha}\|v - \bullet\,\|_{\mathfrak{s}}^{\beta-\zeta} + \|u - \bullet\,\|_{\mathfrak{s}}^{\alpha-\zeta}\|v - \bullet\,\|_{\mathfrak{s}}^{\beta}).$$

From this inequality, we have

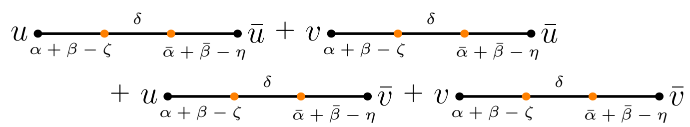
(4.8)



$$=: \|u - v\|_{\mathfrak{s}}^{\zeta}\|\bar{u} - \bar{v}\|_{\mathfrak{s}}^{\eta}\sum_{i=1}^{4} I_i(u, v; \bar{u}, \bar{v}).$$

From Lemma 4.30, we have

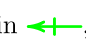
$I_1(u, v; \bar{u}, \bar{v}) \lesssim$



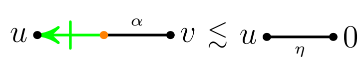
They are bounded from Lemma 4.15, since

$$\alpha + \beta - \zeta, \bar{\alpha} + \bar{\beta} - \eta, \delta \in (-3, 0), \quad (\alpha + \beta - \zeta) + \delta + (\bar{\alpha} + \bar{\beta} - \eta) + 6 > 0.$$

(See Remark 4.29) Therefore we obtain the required bound. $\square$

In order to bound graphs which contain , we need the following lemmas.

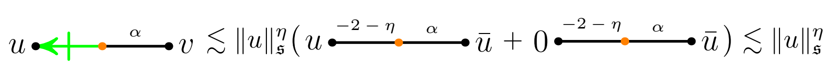
Lemma 4.32. *Let $\alpha \in (-1, 0)$ and $\eta \in (0, 1 + \alpha)$. Then we have*



proof. From Lemma 4.25, we have

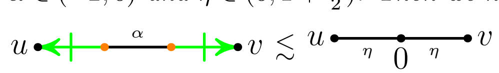
$$|K'(u - \bullet\,) - K'(- \bullet\,)| \lesssim \|u\|_{\mathfrak{s}}^{\eta}(\|u - \bullet\,\|_{\mathfrak{s}}^{-2-\eta} + \| - \bullet\,\|_{\mathfrak{s}}^{-2-\eta}).$$

From Lemma 4.15, we have

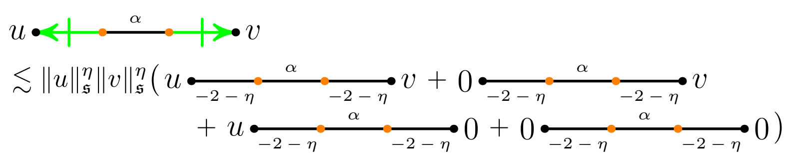


since $(-2 - \eta) + \alpha > -3$. $\square$

Lemma 4.33. *Let $\alpha \in (-2, 0)$ and $\eta \in (0, 1 + \frac{\alpha}{2})$. Then we have*



proof. From Lemma 4.25, we have

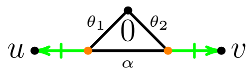
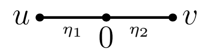


The last four graphs are bounded since $(-2 - \eta) + \alpha + (-2 - \eta) + 6 > 0$. $\square$

Lemma 4.34. *Let $\alpha \in (-2,0)$, $\theta_1,\theta_2>0$, and $\eta_1,\eta_2\in(0,1)$. Assume that*

$$2+\alpha+\theta_1\vee\theta_2<\eta_1+\eta_2<2+\alpha+\theta_1+\theta_2.$$

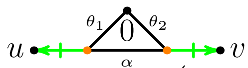
Then we have

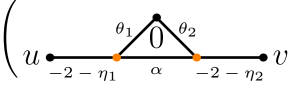
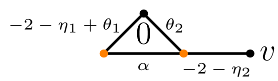
 $\lesssim$  $\times R_{\theta_1,\theta_2}(u,v),$

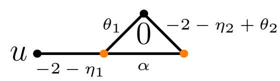
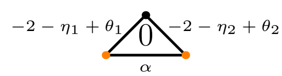
locally uniformly over u,v. Here,

$$R_{\theta_1,\theta_2}(u;v)=\|u\|_{\mathfrak{s}}^{\theta_1}\|v\|_{\mathfrak{s}}^{\theta_2}\|u-v\|_{\mathfrak{s}}^{-\theta_1-\theta_2}+\|u\|_{\mathfrak{s}}^{\theta_1}\|u-v\|_{\mathfrak{s}}^{-\theta_1}+\|v\|_{\mathfrak{s}}^{\theta_2}\|u-v\|_{\mathfrak{s}}^{-\theta_2}+1.$$

proof. From Lemma 4.25, we have

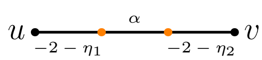
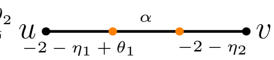


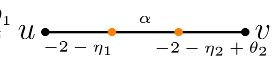
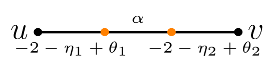
$\lesssim \|u\|_{\mathfrak{s}}^{\eta_1}\|v\|_{\mathfrak{s}}^{\eta_2}\Big($  $+$ 

$+$  $+$  $\Big)$

$=: \|u\|_{\mathfrak{s}}^{\eta_1}\|v\|_{\mathfrak{s}}^{\eta_2}(I_1(u,v)+I_2(v)+I_3(u)+I_4).$

Since $\|0-\bullet\|_{\mathfrak{s}}^{\theta_1}\lesssim\|u\|_{\mathfrak{s}}^{\theta_1}+\|u-\bullet\|_{\mathfrak{s}}^{\theta_1}$, we have

$I_1(u,v)\lesssim\|u\|_{\mathfrak{s}}^{\theta_1}\|v\|_{\mathfrak{s}}^{\theta_2}$  $+\|v\|_{\mathfrak{s}}^{\theta_2}$ 

$+\|u\|_{\mathfrak{s}}^{\theta_1}$  $+$ 

$$\lesssim\|u\|_{\mathfrak{s}}^{\theta_1}\|v\|_{\mathfrak{s}}^{\theta_2}\|u-v\|_{\mathfrak{s}}^{2-\eta_1-\eta_2+\alpha}+\|v\|_{\mathfrak{s}}^{\theta_2}\|u-v\|_{\mathfrak{s}}^{2-\eta_1-\eta_2+\alpha+\theta_1}+\|u\|_{\mathfrak{s}}^{\theta_1}\|u-v\|_{\mathfrak{s}}^{2-\eta_1-\eta_2+\alpha+\theta_2}+1.$$

Since $2-\eta_1-\eta_2+\alpha>-\theta_1-\theta_2$, locally uniformly we have

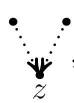
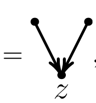
$$I_1(u,v)\lesssim R_{\theta_1,\theta_2}(u,v).$$

We also obtain $I_2,I_3,I_4\lesssim 1$ by similar ways. $\square$

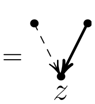
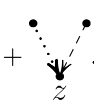
4.8. **Estimates of $\mathcal{P}_l^{(\epsilon,k)}(\tau)(z;\bar z)$.** The estimate of $\mathcal{P}_l^{(\epsilon,k)}(\tau)(z;\bar z)$ is obtained similarly to that of $\mathcal{P}_l^{(k)}(\tau)(z;\bar z)$. The graphical representation of $\delta\hat{\mathcal{W}}_l^{(\epsilon,k)}(\tau)(z)$ is a sum of graphs such that

- One of the edges is replaced by $\dashleftarrow = (\longleftarrow - \leftarrow\cdots)$, ${\color{magenta}\dashleftarrow} = ({\color{magenta}\longleftarrow} - {\color{magenta}\leftarrow\cdots})$ or ${\color{blue}\dashleftarrow} = ({\color{blue}\longleftarrow} - {\color{blue}\leftarrow\cdots})$, instead of $\longleftarrow$, ${\color{magenta}\longleftarrow}$ or ${\color{blue}\longleftarrow}$.
- Some of the edges are replaced by $\leftarrow\cdots$, ${\color{magenta}\leftarrow\cdots}$, or ${\color{blue}\leftarrow\cdots}$, instead of $\longleftarrow$, ${\color{magenta}\longleftarrow}$ or ${\color{blue}\longleftarrow}$.

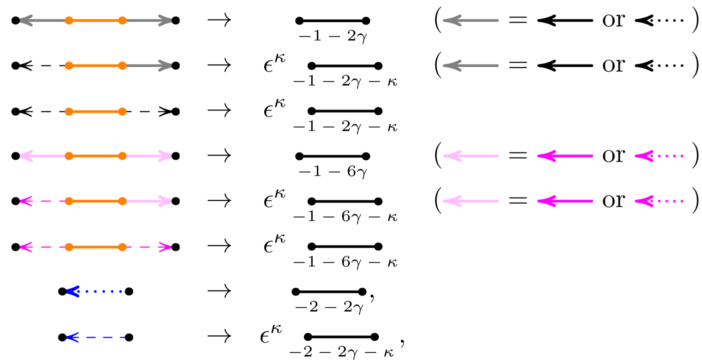
For example, since

$\hat{\mathcal{W}}^{(\epsilon,2)}($$)(z;\cdot,\cdot)=$ , $\quad\hat{\mathcal{W}}^{(2)}($$)(z;\cdot,\cdot)=$ ,

we have

$\hat{\mathcal{W}}^{(\epsilon,2)}($$)(z;\cdot,\cdot)-\hat{\mathcal{W}}^{(2)}($$)(z;\cdot,\cdot)=$  $+$ .

Similarly to before, each edge of graphs $\mathcal{P}_l^{(\epsilon,k)}(\tau)(z;\bar{z})$ has the label:



for any small $\kappa>0$. See Subsection 4.6.

Therefore the labelled graph of $\mathcal{P}_l^{(\epsilon,k)}(\tau)(z;\bar{z})$ is obtained by ϵ^κ times that of $\mathcal{P}_l^{(k)}(\tau)(z;\bar{z})$, whose labels are instead κ smaller. Hence the bound of $\mathcal{P}_l^{(\epsilon,k)}(\tau)(z;\bar{z})$ is obtained similarly to that of $\mathcal{P}_l^{(k)}(\tau)(z;\bar{z})$.

4.9. **Notes on the case $\gamma\ge\frac14$.** In order to obtain the required estimates as in Theorem 4.13, all of labels in the graphs must be greater than -3. If $\gamma\ge\frac14$, we cannot define the limit of $\hat{\Pi}_0^{(\epsilon)}$ by the renormalization described above. From the definition of , we have

$$(\Pi_0^{(\epsilon)})(z)=I_2(\mathcal{W}^{(\epsilon,2)}()(z))+\mathcal{W}^{(\epsilon,0)}()(z),$$

where

$$\mathcal{W}^{(\epsilon,2)}()(z;w_1,w_2)=\pi\partial_x^\gamma K_\epsilon'(z-w_1)\pi\partial_x^\gamma K_\epsilon'(z-w_2),$$
$$\mathcal{W}^{(\epsilon,0)}()(z)=\int_{w\in\mathbb{R}\times[0,1)}(\pi\partial_x^\gamma K_\epsilon'(z-w))^2dw.$$

Since the constant term diverges, we define $C^{(\epsilon)}=\mathcal{W}^{(\epsilon,0)}()(z)$ and

$$(\hat{\Pi}_0^{(\epsilon)})(z)=I_2(\hat{\mathcal{W}}^{(\epsilon,2)}()(z)),\quad \hat{\mathcal{W}}^{(\epsilon,2)}()(z)=\mathcal{W}^{(\epsilon,2)}()(z).$$

If $\gamma<\frac14$, we can define $\hat{\Pi}_0(z)$ by

$$(\hat{\Pi}_0)(z)=I_2(\hat{\mathcal{W}}^{(2)}()(z;\cdot,\cdot)),$$
$$\hat{\mathcal{W}}^{(2)}()(z;w_1,w_2)=\pi\partial_x^\gamma K'(z-w_1)\pi\partial_x^\gamma K'(z-w_2),$$

because we have the bound

$$\mathcal{P}^{(2)}()(z;\bar{z})\lesssim\|z-\bar{z}\|_{\mathfrak{s}}^{-2-4\gamma}$$

and $-2-4\gamma>-3$. See Section 5.

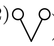
On the other hand, if $\gamma\ge\frac14$ we cannot use this estimate. In fact, for every test function φ we obtain the following estimate.

$$(\hat{\Pi}_0^{(\epsilon,2)})(\varphi)\sim\begin{cases}|\log\epsilon| & \gamma=\frac14\\ \epsilon^{1-4\gamma} & \gamma>\frac14.\end{cases}$$

Now we prove it. Let $G_\epsilon = G * \rho_\epsilon$. In $z, \bar z \in B_{\mathfrak{s}}(0, \frac{1}{2})$, $\mathcal{P}^{(\epsilon,2)}($$)(z;\bar z)$ is given by

$\mathcal{P}^{(\epsilon,2)}($$)(z;\bar z)$
$$\sim \int_{w_1,w_2\in\mathbb{R}\times[0,1)} \pi\partial_x^\gamma G'_\epsilon(z-w_1)\pi\partial_x^\gamma G'_\epsilon(z-w_2) \times \pi\partial_x^\gamma G'_\epsilon(\bar z-w_1)\pi\partial_x^\gamma G'_\epsilon(\bar z-w_2)dw_1dw_2$$
$$\sim (\partial_x^{2\gamma} G * \rho_\epsilon * \overleftarrow{\rho_\epsilon})(|t-\bar t|, x-\bar x)^2 =: I_\epsilon(z-\bar z),$$

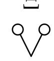
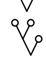
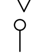
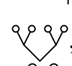
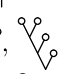
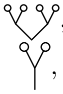
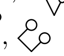
since $G' * \overleftarrow{G'}(t,x) = -\frac{1}{2}G(|t|,x)$. From the scaling property of G, we have $I_\epsilon(t,x) = \epsilon^{-2-4\gamma}I_1(\epsilon^{-2}t, \epsilon^{-1}x)$. Then we have

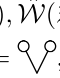
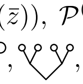
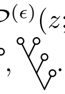
$(\hat\Pi_0^{(\epsilon,2)}$$)(\varphi)$
$$\sim \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(z)\varphi(\bar z)I_\epsilon(z-\bar z)dzd\bar z$$
$$= \epsilon^{1-4\gamma}\int_{\mathbb{R}^2}\epsilon^3\varphi(\epsilon^2 t, \epsilon x)\left(\int_{\mathbb{R}^2}\varphi(\epsilon^2\bar t, \epsilon\bar x)I_1(t-\bar t, x-\bar x)d\bar t d\bar x\right)dtdx$$
$$\sim \epsilon^{1-4\gamma}\int_{\|z\|_{\mathfrak{s}}\lesssim\epsilon^{-1}} I_1(z)dz.$$

Since there exists $\psi \in \mathcal{S}(\mathbb{R})$ such that
$$I_1(t,x) \sim \frac{1}{t^{1+2\gamma}}\left(\partial_x^{2\gamma}\psi\left(\frac{x}{\sqrt{t}}\right)\right)^2$$
for $\|z\|_{\mathfrak{s}} > 1$, we have
$$\int_{\|z\|_{\mathfrak{s}}\lesssim\epsilon^{-1}} I_1(z)dz \sim \int_{|t|\lesssim\epsilon^{-2}} 1\wedge t^{-\frac{1}{2}-2\gamma}dt \sim \begin{cases} |\log\epsilon| & \epsilon = \frac{1}{4} \\ 1 & \epsilon > \frac{1}{4}. \end{cases}$$

5. Renormalization in $0 \le \gamma < \frac{1}{10}$

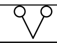
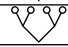
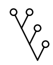
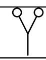
Let $0 \le \gamma < \frac{1}{10}$. We choose sufficiently small $\kappa > 0$ and set $\alpha_0 = -\frac{3}{2} - \gamma - \kappa$. Then all elements in $\mathcal{F}_-$ are as follows.

Homogeneity	Symbol
$-\frac{3}{2}-\gamma-\kappa$	Ξ
$-1-2\gamma-2\kappa$	
$-\frac{1}{2}-3\gamma-3\kappa$	
$-\frac{1}{2}-\gamma-\kappa$	
$-4\gamma-4\kappa$	, 
$-2\gamma-2\kappa$	, 
0	$\mathbf{1}$

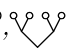
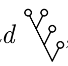
In this section, we obtain the bounds of $\mathcal{P}(z;\bar z) = (\hat{\mathcal{W}}(z), \hat{\mathcal{W}}(\bar z))$, $\mathcal{P}^{(\epsilon)}(z;\bar z) = (\hat{\mathcal{W}}(z) - \hat{\mathcal{W}}^{(\epsilon)}(z), \hat{\mathcal{W}}(\bar z) - \hat{\mathcal{W}}^{(\epsilon)}(\bar z))$ and constants $C_\tau^{(\epsilon)}$ for $\tau =$ , , .

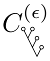
Proposition 5.1. *For each $\tau \in \mathcal{F}_-$, we have the bounds of $\mathcal{P}(z;\bar z)$ and $\mathcal{P}^{(\epsilon)}(z;\bar z)$ as follows.*

Symbol	$\mathcal{P}(z;\bar z)$	$\mathcal{P}^{(\epsilon)}(z;\bar z)$

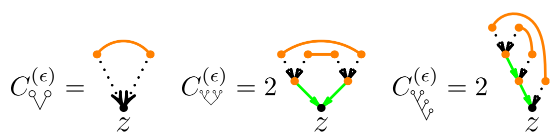
	$\Vert z-\bar z\Vert_{\mathfrak s}^{-2-4\gamma}$	$\epsilon^{\theta}\Vert z-\bar z\Vert_{\mathfrak s}^{-2-4\gamma-\theta}$
	$\Vert z-\bar z\Vert_{\mathfrak s}^{-1-6\gamma}$	$\epsilon^{\theta}\Vert z-\bar z\Vert_{\mathfrak s}^{-1-6\gamma-\theta}$
	$\Vert z-\bar z\Vert_{\mathfrak s}^{-1-2\gamma}$	$\epsilon^{\theta}\Vert z-\bar z\Vert_{\mathfrak s}^{-1-2\gamma-\theta}$
	$\Vert z-\bar z\Vert_{\mathfrak s}^{-8\gamma}$	$\epsilon^{\theta}\Vert z-\bar z\Vert_{\mathfrak s}^{-8\gamma-\theta}$
	$\Vert z\Vert_{\mathfrak s}^{\frac12-3\gamma-\theta}\Vert \bar z\Vert_{\mathfrak s}^{\frac12-3\gamma-\theta}$ $\times\Vert z-\bar z\Vert_{\mathfrak s}^{-1-2\gamma}$, $\Vert z-\bar z\Vert_{\mathfrak s}^{-8\gamma}$, $\Vert z\Vert_{\mathfrak s}^{-4\gamma-\theta}\Vert \bar z\Vert_{\mathfrak s}^{-4\gamma-\theta}$	$\epsilon^{\theta}\Vert z\Vert_{\mathfrak s}^{\frac12-3\gamma-\theta_1}\Vert \bar z\Vert_{\mathfrak s}^{\frac12-3\gamma-\theta_1}$ $\times\Vert z-\bar z\Vert_{\mathfrak s}^{-1-2\gamma-\theta_2}$, $\epsilon^{\theta}\Vert z-\bar z\Vert_{\mathfrak s}^{-8\gamma-\theta}$, $\epsilon^{\theta}\Vert z\Vert_{\mathfrak s}^{-4\gamma-\theta_1}\Vert \bar z\Vert_{\mathfrak s}^{-4\gamma-\theta_1}$
	$\Vert z-\bar z\Vert_{\mathfrak s}^{-4\gamma}$	$\epsilon^{\theta}\Vert z-\bar z\Vert_{\mathfrak s}^{-4\gamma-\theta}$
	$\Vert z\Vert_{\mathfrak s}^{\frac12-\gamma-\theta}\Vert \bar z\Vert_{\mathfrak s}^{\frac12-\gamma-\theta}$ $\times\Vert z-\bar z\Vert_{\mathfrak s}^{-1-2\gamma}$, $\Vert z\Vert_{\mathfrak s}^{-2\gamma}\Vert \bar z\Vert_{\mathfrak s}^{-2\gamma}$	$\epsilon^{\theta}\Vert z\Vert_{\mathfrak s}^{\frac12-\gamma-\theta_1}\Vert \bar z\Vert_{\mathfrak s}^{\frac12-\gamma-\theta_1}$ $\times\Vert z-\bar z\Vert_{\mathfrak s}^{-1-2\gamma-\theta_2}$, $\epsilon^{\theta}\Vert z\Vert_{\mathfrak s}^{-2\gamma-\theta}\Vert \bar z\Vert_{\mathfrak s}^{-2\gamma-\theta}$

Here $\theta,\theta_1,\theta_2>0$ are any sufficiently small numbers.

Proposition 5.2. *For* ,  *and* , *we have*

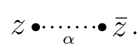
$C^{(\epsilon)}$  $\sim \epsilon^{-1-2\gamma}$, $|C^{(\epsilon)}$  $|\lesssim \epsilon^{-4\gamma}$, $|C^{(\epsilon)}$  $|\lesssim \epsilon^{-4\gamma}$.

5.1. **Values of constants.** We define constants $C_\tau^{(\epsilon)}$ as follows.

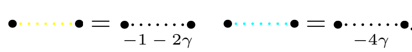


Here, factors 2 come from the symmetry of graphs.

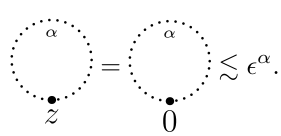
For a family of functions $\{A^{(\epsilon)}\}_{\epsilon\in(0,1]}$ on $\mathbb{R}^2$ such that $\operatorname{supp} A^{(\epsilon)}\subset B_{\mathfrak s}(0,1)$ and $\sup_{z,\epsilon}(\Vert z\Vert_{\mathfrak s}+\epsilon)^{-\alpha}|A^{(\epsilon)}(z)|<\infty$, we write

$A^{(\epsilon)}(z-\bar z)=$ 

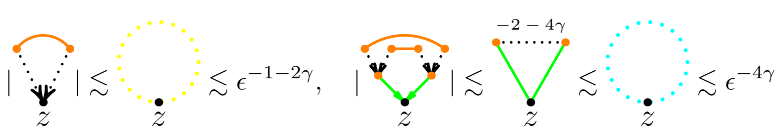
For simplicity, we write

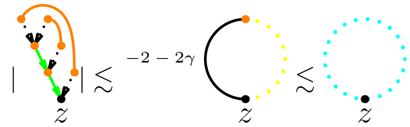


We can use similar contractions to those in Subsection 4.7. Our goal is to obtain graphs as follows.



Now we can prove Proposition 5.2 as follows.



 $\lesssim \epsilon^{-4\gamma}.$

$C^{(\epsilon)}$ has more precise estimate. Since $\partial_x^\gamma G' = \partial_x^\gamma K' + \partial_x^\gamma R'$ and $|\partial_x^\gamma R'(t,x)| \lesssim (1+|x|)^{-2-\gamma}$ uniformly over t from Remark 4.18, we have

$$C^{(\epsilon)} = \int_{w\in\mathbb{R}\times[0,1)} (\pi\partial_x^\gamma K'_\epsilon(-w))^2 dw \sim \int_{w\in\mathbb{R}\times[0,1)} (\pi\partial_x^\gamma G'_\epsilon(-w))^2 dw.$$

Lemma 5.3. *Let f be a function on $\mathbb{R}^2$ such that $|f(t,x)| \lesssim_t (1+|x|)^{-2}$ for every $t\in\mathbb{R}$ and $\int_{\mathbb{R}\times[0,1)}(\pi f)^2(z)dz < \infty$. Let $f_\epsilon(t,x) = f(\epsilon^{-2}t, \epsilon^{-1}x)$. Then we have*

$$\int_{\mathbb{R}\times[0,1)} (\pi f_\epsilon)^2(z)dz \sim \epsilon^3.$$

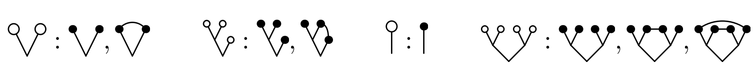
proof. For such f, we have

$$\begin{aligned}\int_{\mathbb{R}\times[0,1)} (\pi f_\epsilon)^2(z)dz &= \int_{\mathbb{R}} dt \int_0^1 \left(\sum_{n=-\infty}^{\infty} f(\epsilon^{-2}t, \epsilon^{-1}(x+n))\right)^2 dx \\ &= \int_{\mathbb{R}} dt \sum_{k=-\infty}^{\infty} \mathcal{F}(f(\epsilon^{-2}t,\epsilon^{-1}\cdot))(k)^2 \\ &= \epsilon^3 \int_{\mathbb{R}} dt\ \epsilon \sum_{k=-\infty}^{\infty} \mathcal{F}f(t,\cdot)(\epsilon k)^2 \sim \epsilon^3 \int_{\mathbb{R}^2} \mathcal{F}f(t,x)^2 dtdx.\end{aligned}$$

□

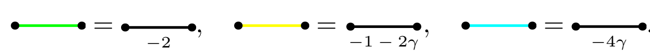
Since $\partial_x^\gamma G'_\epsilon(t,x) = \epsilon^{-2-\gamma}(\partial_x^\gamma G' * \rho)(\epsilon^{-2}t,\epsilon^{-1}x)$ and $|(\partial_x^\gamma G' * \rho)(z)| \lesssim (1+\|z\|_{\mathfrak{s}})^{-2-\gamma}$ from Lemma 4.17 (and from Lemma 10.17 of [3]), we have $C^{(\epsilon)} \sim \epsilon^{-1-2\gamma}$.

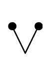
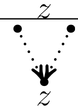
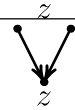
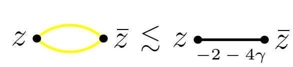
5.2. **Bounds of , , , and .** We start with these five elements. The possible contractions are as follows.

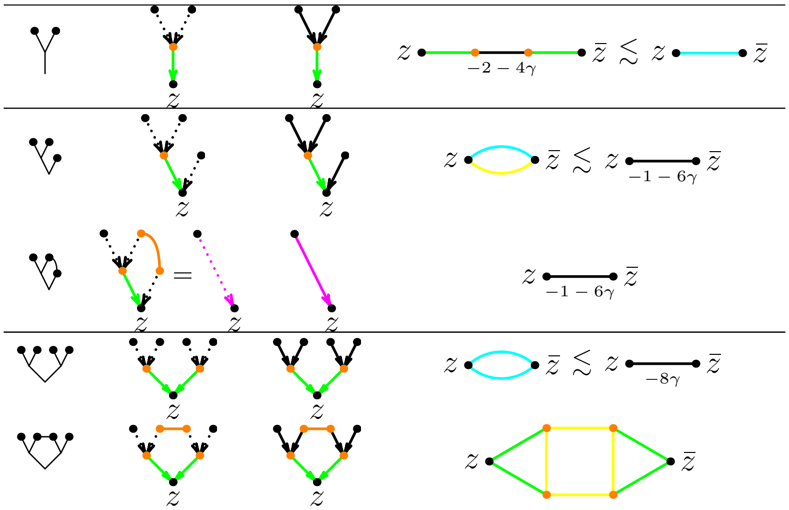


Here contractions containing  are omitted, because they vanish due to the fact that $\int K' = 0$.

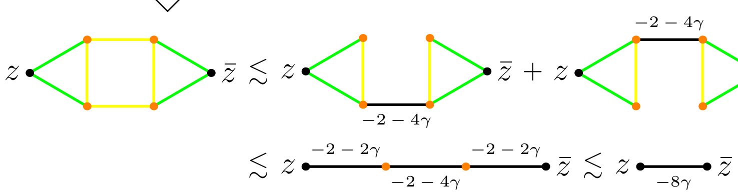
For each graph except constants, $\hat{\mathcal{W}}^{(\epsilon)}(z)$, $\hat{\mathcal{W}}(z)$ and $\mathcal{P}(z;\bar{z}) = (\hat{\mathcal{W}}(z), \hat{\mathcal{W}}(\bar{z}))$ are shown in the following table. For simplicity., we write



	$\hat{\mathcal{W}}^{(\epsilon)}(z)$	$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$
			
			

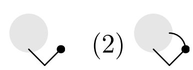


The estimate of is obtained from Lemma 4.30 as follows.



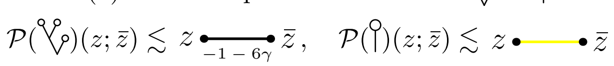
5.3. **Bounds of and .** The estimates of these two elements are more complicated since their graphs have edges written by . We classify all contractions into two types:

(5.1) (1) (2)

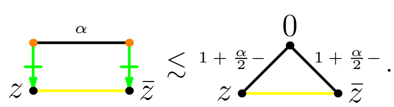


Here represents every contraction of or .

(1) The bounds of (1) are consequences of those of or . Since



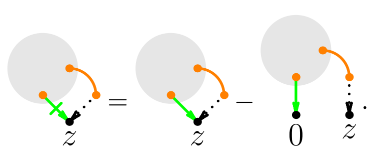
the bounds of $\mathcal{P}(\tau)(z;\bar{z})$ are obtained from Lemma 4.33 as follows.



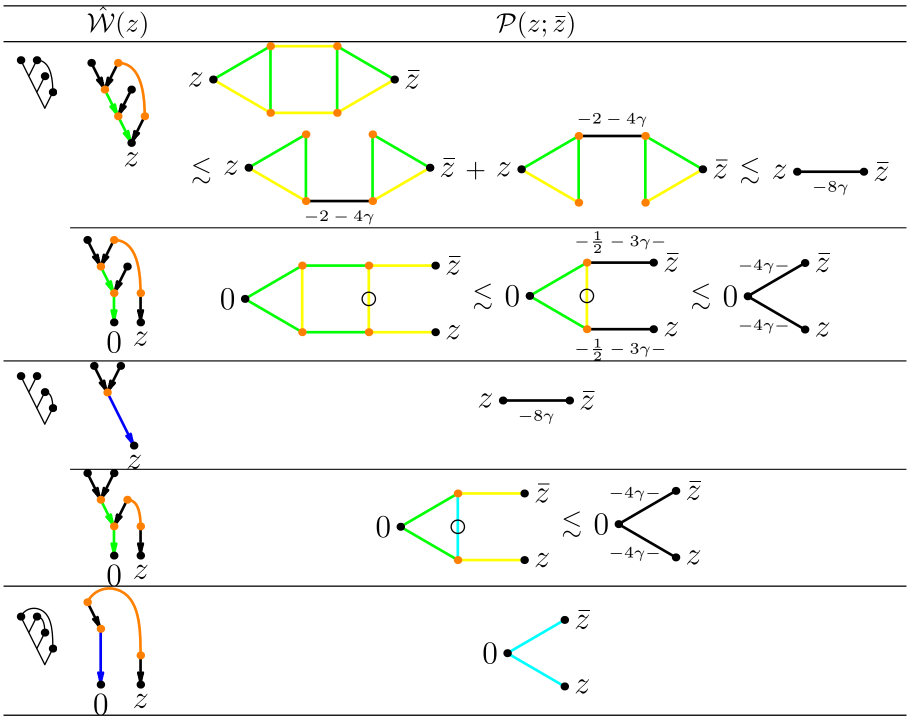
Here $\alpha = -1-6\gamma$ ( ), $-1-2\gamma$ ( ). The label "$\zeta-$" denotes that it can be replaced by $\zeta-\kappa$ for every $\kappa>0$.

(2) We decompose graphs at the edge as follows.

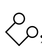
(5.2)

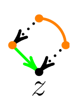


The bounds of (2) are obtained from Lemma 4.31 and shown in the next tables. We mark an edge where we use Lemma 4.31 by a circle (○).



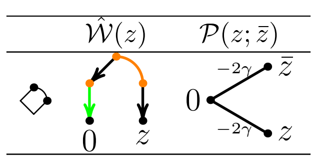
Remark 5.4. *We precisely describe the application of Lemma 4.31 for the fourth one. Let $\alpha = \bar{\alpha} = -2$, $\beta = \bar{\beta} = -1-2\gamma$, $\delta = -4\gamma$. Since $\alpha+\bar{\alpha}+\beta+\bar{\beta}+\delta+6 = -8\gamma$ and $\alpha \vee \bar{\alpha} \vee \beta \vee \bar{\beta} < -4\gamma$, we can set $\zeta = \eta = -4\gamma - \kappa$ for small $\kappa > 0$.*

For , since



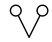
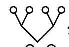
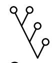
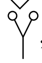
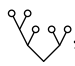
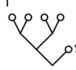
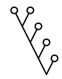
$$= \int K'(z) L_\epsilon^{(2)}(z) dz = 0$$

because K' is odd and $L_\epsilon^{(2)}$ is even in x, we consider the second term.

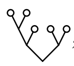
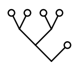
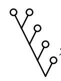


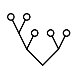
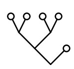
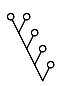
6. Renormalization in $\frac{1}{10} \leq \gamma < \frac{1}{6}$

Let $\frac{1}{10} \leq \gamma < \frac{1}{6}$. Then all elements in $\mathcal{F}_-$ are as follows.

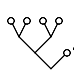
Homogeneity	Symbol
$-\frac{3}{2}-\gamma-\kappa$	Ξ
$-1-2\gamma-2\kappa$	
$-\frac{1}{2}-3\gamma-3\kappa$	
$-\frac{1}{2}-\gamma-\kappa$	
$-4\gamma-4\kappa$	, 
$-2\gamma-2\kappa$	, 
$\frac{1}{2}-5\gamma-5\kappa$	, , 
0	$\mathbf{1}$

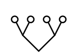
It remains to obtain the bounds of elements with homogeneity $\frac{1}{2}-5\gamma-5\kappa$. The other bounds are same as in Section 5.

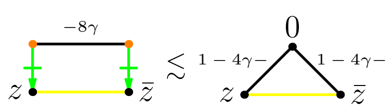
Proposition 6.1. *For* ,  *and* , *we have the bounds of* $\mathcal{P}(z;\bar{z})$ *as follows.*

Symbol	$\mathcal{P}(z;\bar{z})$
	$\\|z\\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta}\\|z-\bar{z}\\|_{\mathfrak{s}}^{-4\gamma}$, $\\|z-\bar{z}\\|_{\mathfrak{s}}^{1-10\gamma-\theta}$, $\\|z\\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}$
	$\\|z\\|_{\mathfrak{s}}^{1-4\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{1-4\gamma-\theta}\\|z-\bar{z}\\|_{\mathfrak{s}}^{-1-2\gamma}$, $\\|z-\bar{z}\\|_{\mathfrak{s}}^{1-10\gamma-\theta}$, $\\|z\\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}$
	$(\\|z\\|_{\mathfrak{s}}^{3-14\gamma-\theta}+\\|\bar{z}\\|_{\mathfrak{s}}^{3-14\gamma-\theta})\\|z-\bar{z}\\|_{\mathfrak{s}}^{-2+4\gamma}$, $\\|z-\bar{z}\\|_{\mathfrak{s}}^{1-10\gamma-\theta}$, $\\|z\\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}$

Here θ is any sufficiently small number. The bounds of $\mathcal{P}^{(\epsilon)}(z;\bar{z})$ are obtained by multiplying ϵ^{θ} to above bounds whose indexes are slightly subtracted as in Proposition 5.1.

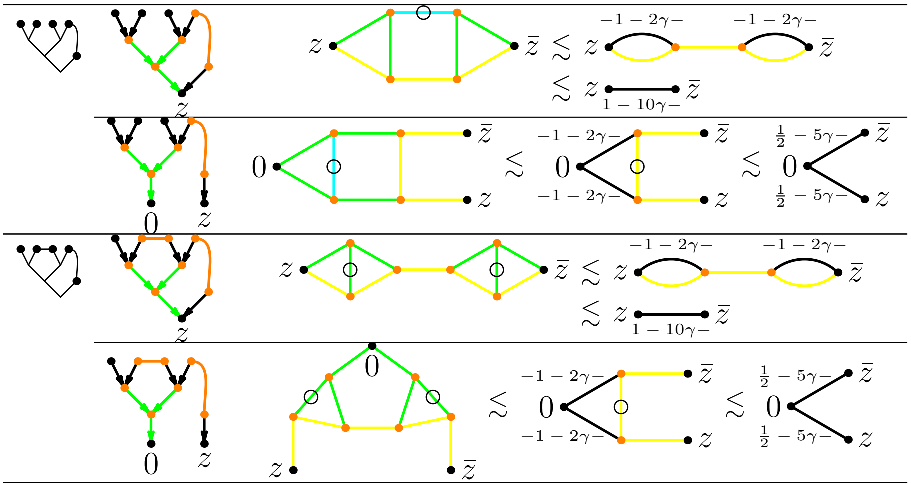
6.1. **Bounds of** **.** We classify all contractions into two types as in (5.1).

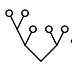
(1) The bounds of (1) are consequences of those of  and obtained from Lemma 4.33 as follows.

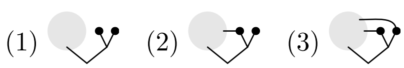


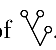
(2) We decompose graphs as in (5.2). The bounds of (2) are obtained from Lemma 4.31 as follows.

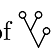
$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$

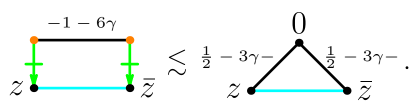


6.2. **Bounds of** **.** We classify all contractions into three types:

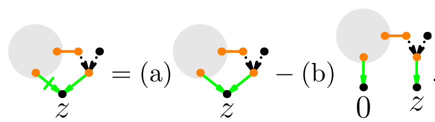
(6.1) 

Here represents every contraction of .

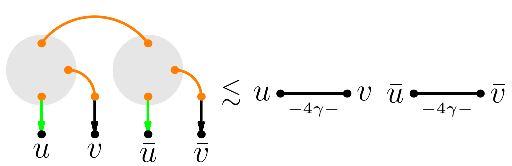
(1) The bounds of (1) are consequences of those of  and obtained from Lemma 4.33 as follows.



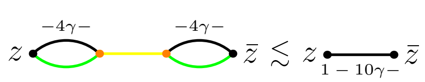
(2) We decompose graphs at the edge as follows.

(6.2) 

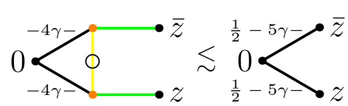
Since the graphs $\mathcal{P}(z;\bar{z})$ contain subgraphs of the shape

(6.3) 

we have for (a)

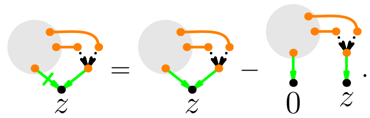


and for (b)

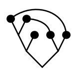
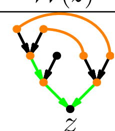
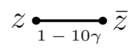
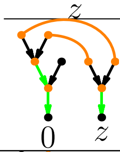
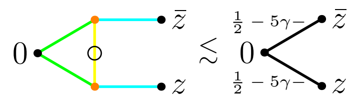
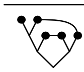
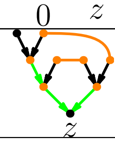
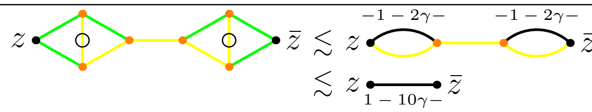
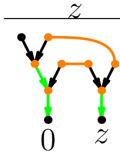
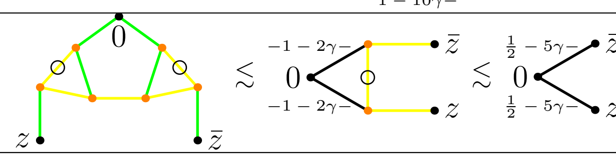


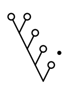
(3) We decompose graphs at the edge  as follows.

(6.4)

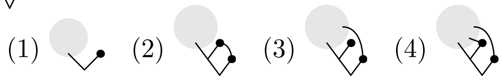


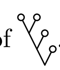
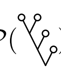
The bounds of (3) are obtained from Lemma 4.31 as follows.

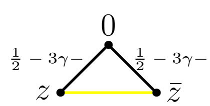
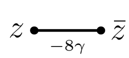
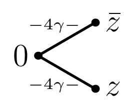
	$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$
		
		
		
		

6.3. **Bounds of** **.** We classify all contractions into four types:

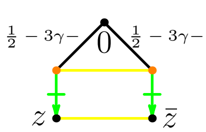
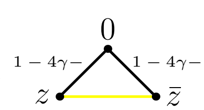
(6.5)

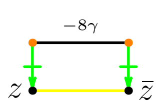
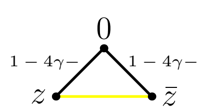


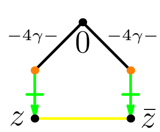
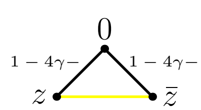
(1) The bounds of (1) are consequences of those of . The bounds of $\mathcal{P}($$)(z;\bar{z})$ are any of the following three graphs.

(A)  (B)  (C) 

For each case, we have

(A)  $\lesssim$  $\times R_{\frac{1}{2}-3\gamma-}(z;\bar{z})$, (Lemma 4.34)

(B)  $\lesssim$ , (Lemma 4.33)

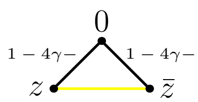
(C)  $\lesssim$ . (Lemma 4.32)

Here we write $R_\theta = R_{\theta,\theta}$ for $\theta > 0$. The first one is a little bit complicated. Since

$$R_{\theta_1,\theta_2}(z;\bar z) \lesssim (\|z\|_{\mathfrak s}^{\theta_1+\theta_2} + \|\bar z\|_{\mathfrak s}^{\theta_1+\theta_2})\|z-\bar z\|_{\mathfrak s}^{-\theta_1-\theta_2} \quad (\theta_1,\theta_2>0)$$

and $\|z\|_{\mathfrak s}^{1-4\gamma-}\|\bar z\|_{\mathfrak s}^{1-4\gamma-} \lesssim \|z\|_{\mathfrak s}^{2-8\gamma-} + \|\bar z\|_{\mathfrak s}^{2-8\gamma-}$, we have

(6.6)



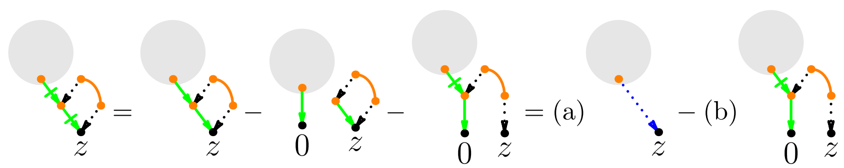
$$\times R_{\frac12-3\gamma-}(z;\bar z)$$

$$\lesssim (\|z\|_{\mathfrak s}^{2-8\gamma-} + \|\bar z\|_{\mathfrak s}^{2-8\gamma-})\|z-\bar z\|_{\mathfrak s}^{-1-2\gamma}(\|z\|_{\mathfrak s}^{1-6\gamma-} + \|\bar z\|_{\mathfrak s}^{1-6\gamma-})\|z-\bar z\|_{\mathfrak s}^{-1+6\gamma}$$

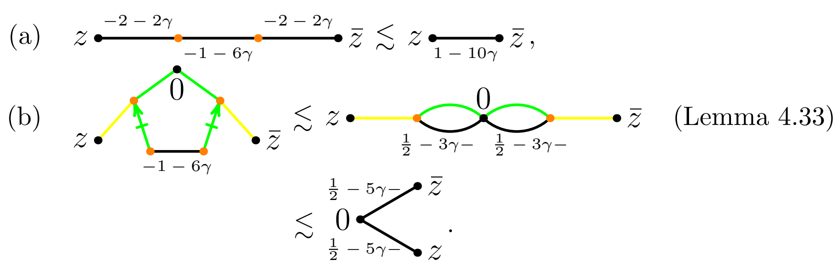
$$\lesssim (\|z\|_{\mathfrak s}^{3-14\gamma-} + \|\bar z\|_{\mathfrak s}^{3-14\gamma-})\|z-\bar z\|_{\mathfrak s}^{-2+4\gamma}.$$

(2) We decompose graphs at edges as follows.

(6.7)

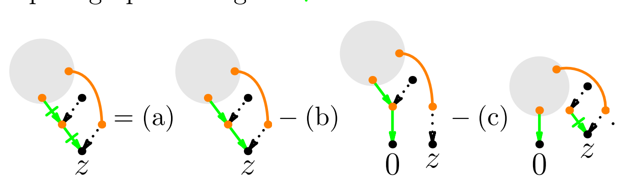


By using the bounds of $\mathcal P$( ) we have

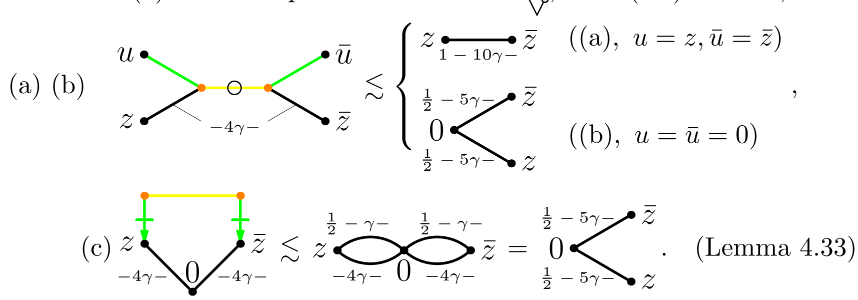


(3) We decompose graphs at edges as follows.

(6.8)

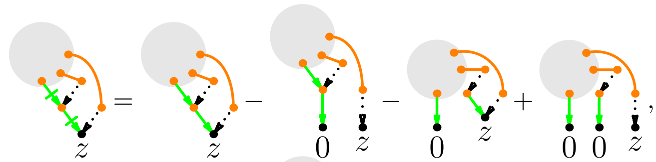


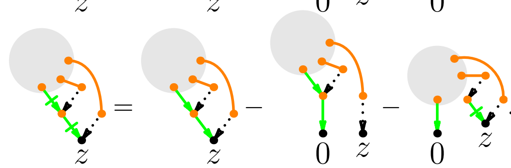
The bounds of (3) are consequences of those of , as in (6.3). In fact,



Here a thin black line in the first graph is not an edge, only to point out the order "$-4\gamma-$" of both edges.

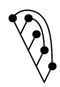
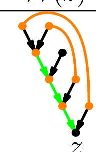
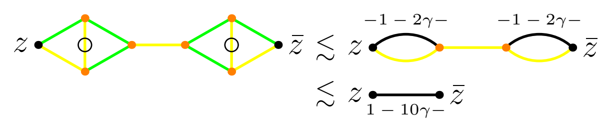
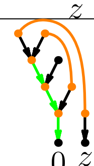
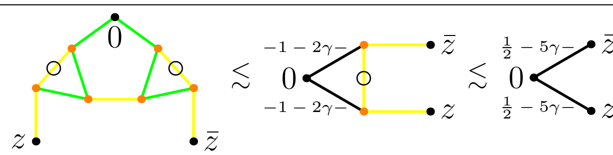
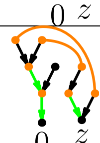
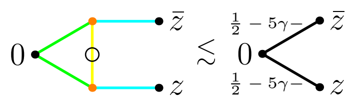
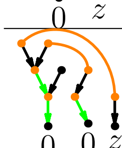
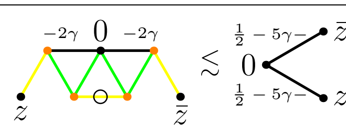
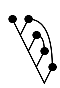
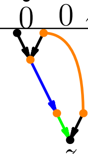
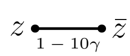
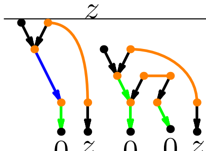
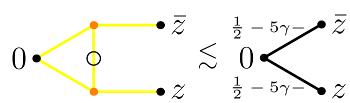
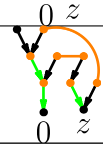
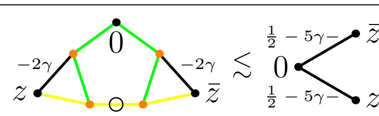
(4) We decompose graphs as any of the following two ways.

(6.9) 

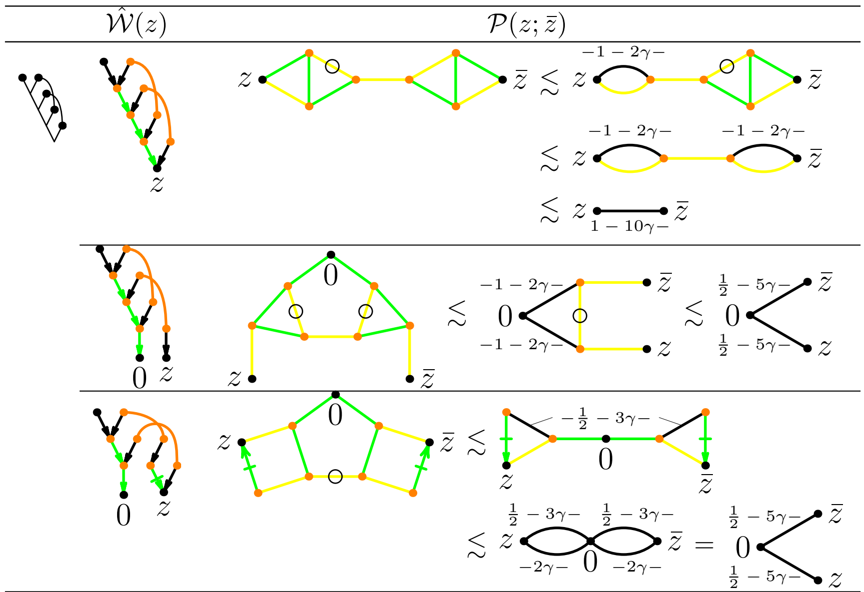
(6.10) 

The bounds of (4) are obtained as follows.

Bounds of the case (6.9)

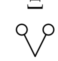
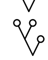
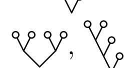
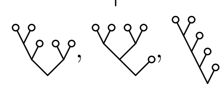
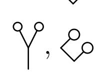
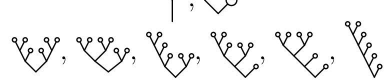
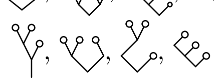
	$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$
		
		
		
		
		
		
		

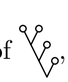
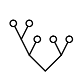
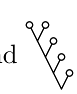
Bounds of the case (6.10)



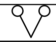
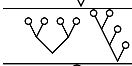
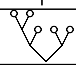
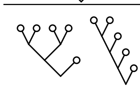
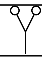
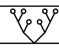
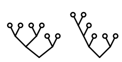
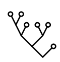
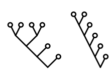
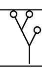
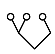
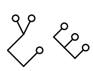
7. Renormalization in $\frac{1}{6} \leq \gamma < \frac{3}{14}$

Let $\frac{1}{6} \leq \gamma < \frac{3}{14}$. Then all elements in $\mathcal{F}_-$ are as follows.

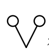
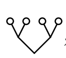
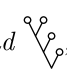
Homogeneity	Symbol
$-\frac{3}{2}-\gamma-\kappa$	Ξ
$-1-2\gamma-2\kappa$	
$-\frac{1}{2}-3\gamma-3\kappa$	
$-4\gamma-4\kappa$	
$-\frac{1}{2}-\gamma-\kappa$	
$\frac{1}{2}-5\gamma-5\kappa$	
$-2\gamma-2\kappa$	
$1-6\gamma-6\kappa$	
$\frac{1}{2}-3\gamma-3\kappa$	
0	$\mathbf{1}$

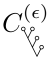
The bounds of ,  and  are different to those in Propositions 5.1 and 6.1 because the numbers of edges  are less than before.

Proposition 7.1. *For each $\tau \in \mathcal{F}_-$, we have the bounds of $\mathcal{P}(z;\bar{z})$ as follows.*

Symbol	$\mathcal{P}(z;\bar{z})$
	$\\|z-\bar{z}\\|_{\mathfrak{s}}^{-2-4\gamma}$
	$\\|z-\bar{z}\\|_{\mathfrak{s}}^{-1-6\gamma}$
	$\\|z-\bar{z}\\|_{\mathfrak{s}}^{-8\gamma}$
	$\\|z-\bar{z}\\|_{\mathfrak{s}}^{-1-2\gamma}$
	$\\|z-\bar{z}\\|_{\mathfrak{s}}^{1-10\gamma-\theta}$
	$\\|z\\|_{\mathfrak{s}}^{1-4\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{1-4\gamma-\theta}\\|z-\bar{z}\\|_{\mathfrak{s}}^{-1-2\gamma}$, $\\|z-\bar{z}\\|_{\mathfrak{s}}^{1-10\gamma-\theta}$, $\\|z\\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{\frac{1}{2}-5\gamma-\theta}$
	$\\|z-\bar{z}\\|_{\mathfrak{s}}^{-4\gamma}$
	$\\|z\\|_{\mathfrak{s}}^{\frac{1}{2}-\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{\frac{1}{2}-\gamma-\theta}\\|z-\bar{z}\\|_{\mathfrak{s}}^{-1-2\gamma}$, $\\|z\\|_{\mathfrak{s}}^{-2\gamma}\\|\bar{z}\\|_{\mathfrak{s}}^{-2\gamma}$
	$\\|z-\bar{z}\\|_{\mathfrak{s}}^{2-12\gamma-\theta}$
	$\\|z\\|_{\mathfrak{s}}^{1-4\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{1-4\gamma-\theta}\\|z-\bar{z}\\|_{\mathfrak{s}}^{-4\gamma}$, $\\|z-\bar{z}\\|_{\mathfrak{s}}^{2-12\gamma-\theta}$, $\\|z\\|_{\mathfrak{s}}^{1-6\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{1-6\gamma-\theta}$
	$\\|z\\|_{\mathfrak{s}}^{\frac{3}{2}-5\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{\frac{3}{2}-5\gamma-\theta}\\|z-\bar{z}\\|_{\mathfrak{s}}^{-1-2\gamma}$, $\\|z-\bar{z}\\|_{\mathfrak{s}}^{2-12\gamma-\theta}$, $\\|z\\|_{\mathfrak{s}}^{1-6\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{1-6\gamma-\theta}$
	$(\\|z\\|_{\mathfrak{s}}^{5-18\gamma-\theta}+\\|\bar{z}\\|_{\mathfrak{s}}^{5-18\gamma-\theta})\\|z-\bar{z}\\|_{\mathfrak{s}}^{-3+6\gamma}$, $\\|z-\bar{z}\\|_{\mathfrak{s}}^{2-12\gamma-\theta}$, $\\|z\\|_{\mathfrak{s}}^{1-6\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{1-6\gamma-\theta}$
	$\\|z-\bar{z}\\|_{\mathfrak{s}}^{1-6\gamma}$
	$\\|z\\|_{\mathfrak{s}}^{\frac{1}{2}-\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{\frac{1}{2}-\gamma-\theta}\\|z-\bar{z}\\|_{\mathfrak{s}}^{-4\gamma}$, $\\|z-\bar{z}\\|_{\mathfrak{s}}^{1-6\gamma}$, $\\|z\\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta}$
	$\\|z\\|_{\mathfrak{s}}^{1-2\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{1-2\gamma-\theta}\\|z-\bar{z}\\|_{\mathfrak{s}}^{-1-2\gamma}$, $\\|z-\bar{z}\\|_{\mathfrak{s}}^{1-6\gamma}$, $\\|z\\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta}\\|\bar{z}\\|_{\mathfrak{s}}^{\frac{1}{2}-3\gamma-\theta}$

Here θ is any sufficiently small number. The bounds of $\mathcal{P}^{(\epsilon)}(z;\bar{z})$ are obtained by multiplying ϵ^θ to above bounds whose indexes are slightly subtracted as in Proposition 5.1.

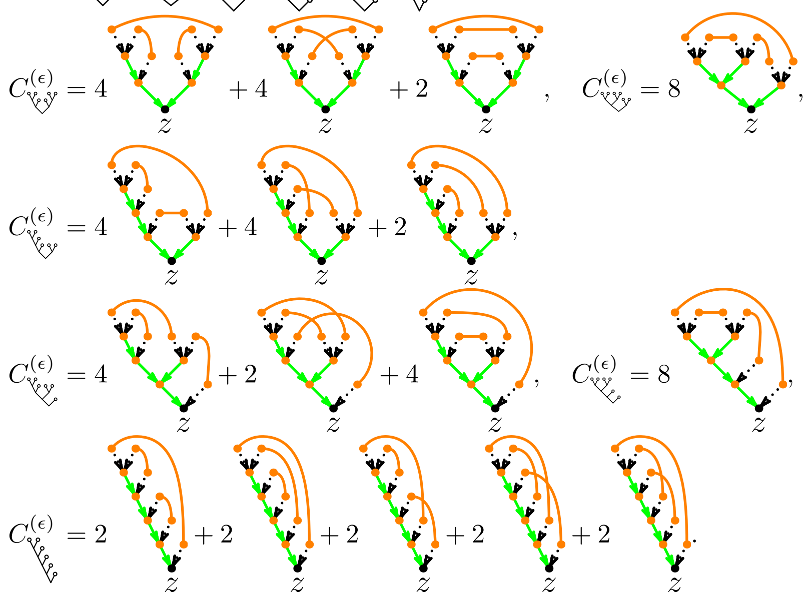
Proposition 7.2. *For* , , *and* , *we have*

$C^{(\epsilon)}$ $\sim \epsilon^{-1-2\gamma}$, $|C^{(\epsilon)}$$| \lesssim \epsilon^{-4\gamma}$, $|C^{(\epsilon)}$$| \lesssim \epsilon^{-4\gamma}$.

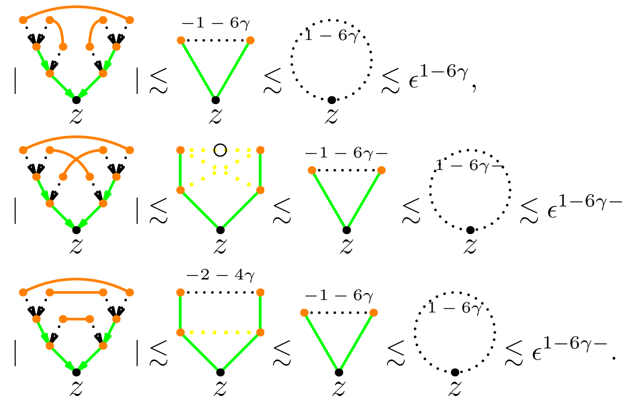
For $\tau \in \mathcal{F}_-$ with $\|\tau\| = 6$ and $\kappa > 0$, we have

$$|C_\tau^{(\epsilon)}| \lesssim \epsilon^{1-6\gamma-\kappa}.$$

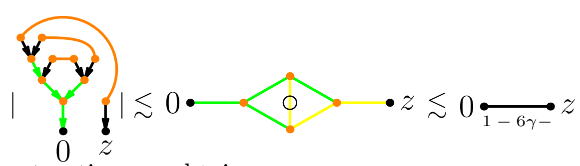
7.1. **Values of constants.** For $\tau =$, , , we define $C_\tau^{(\epsilon)}$ as in Subsection 5.1. For $\tau =$, , , , , , we define



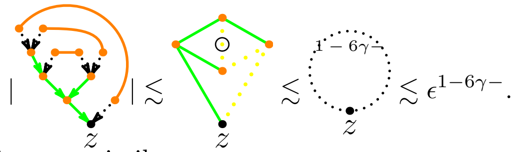
The contractions of these graphs are similar to Subsection 5.1. For , we have



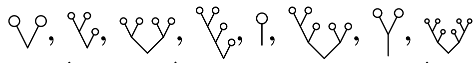
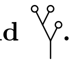
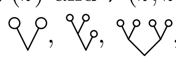
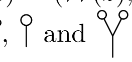
For other τ, the contractions are similar to those of $\mathcal{P}(\tau)(z;\bar{z})$ in Subsection 7.6. For example, since

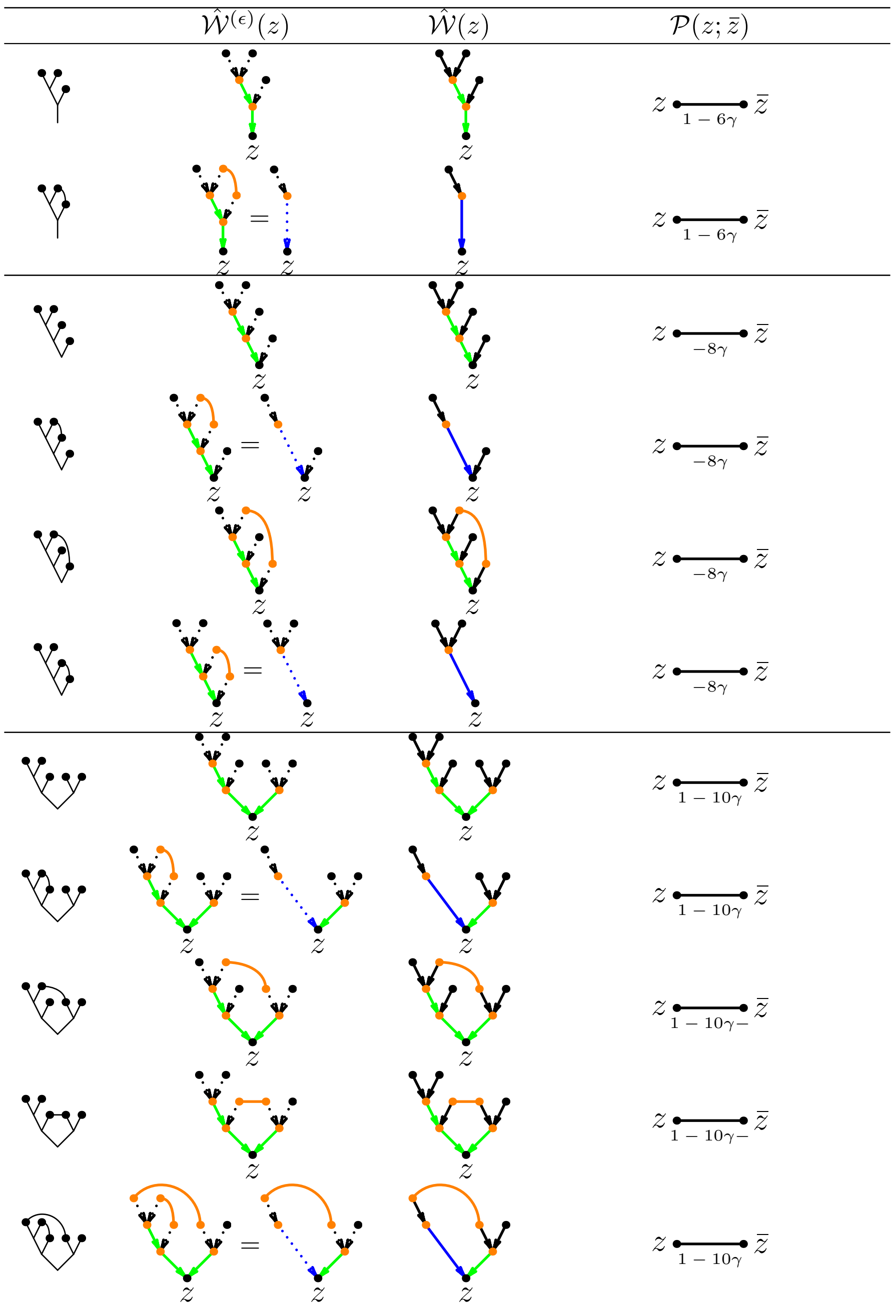


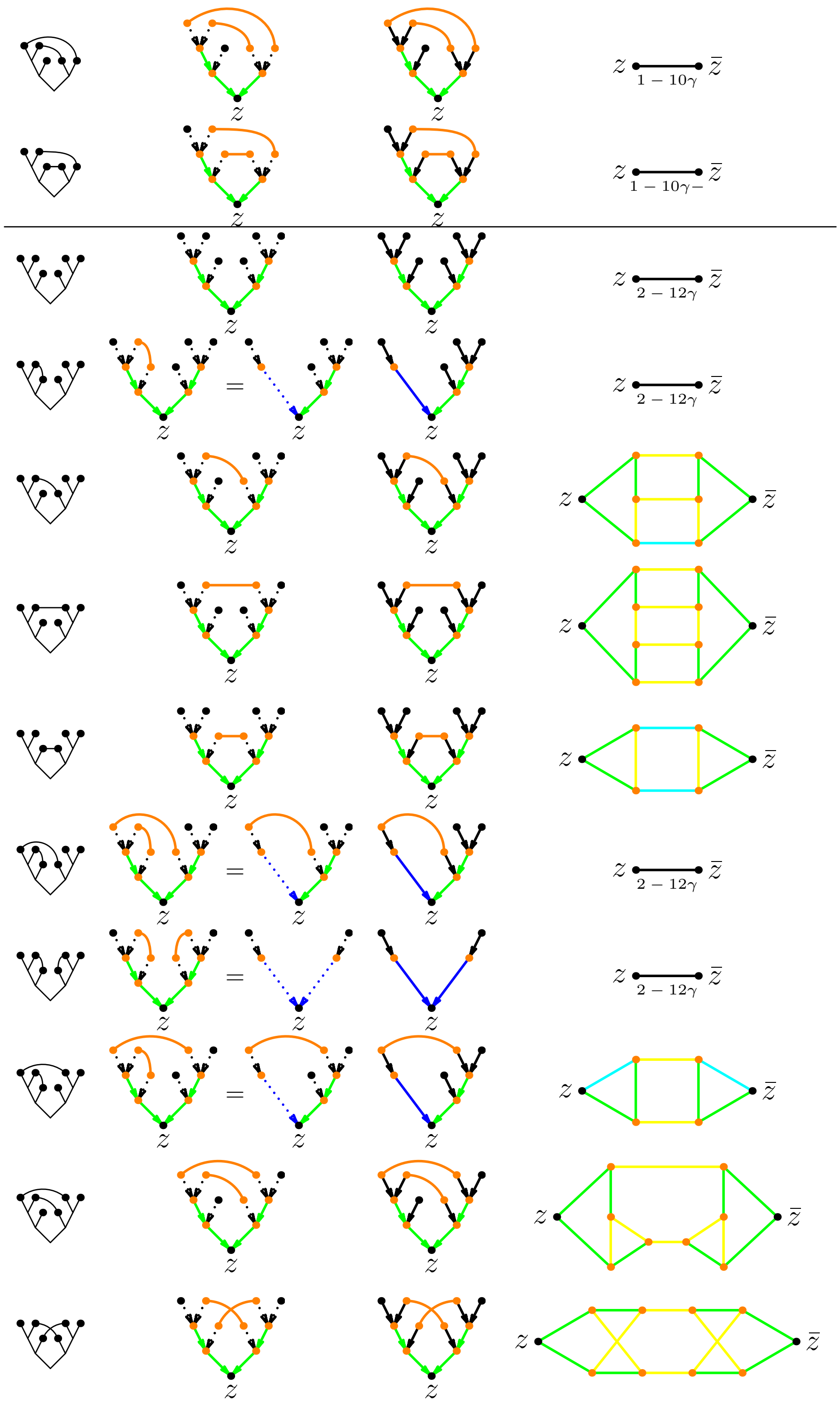
from a similar contraction we obtain

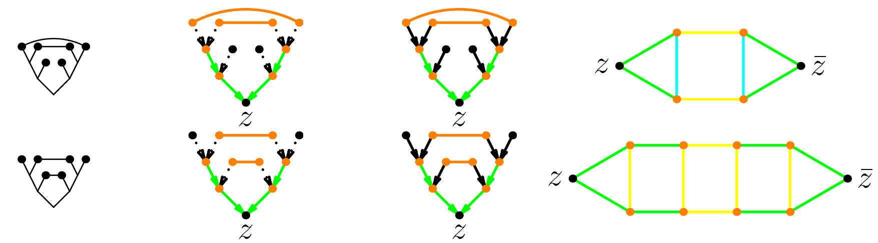


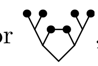
The other calculations are similar.

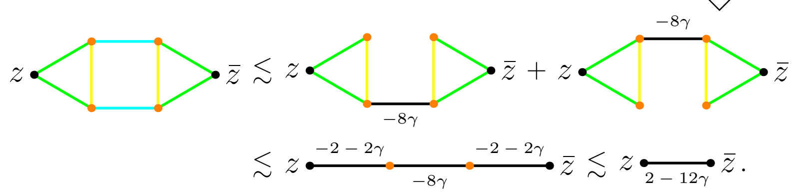
7.2. **Bounds of**  **and** **.** We start with these nine elements. $\hat{\mathcal{W}}^{(\epsilon)}(z)$, $\hat{\mathcal{W}}(z)$ and $\mathcal{P}(z;\bar{z}) = (\hat{\mathcal{W}}(z), \hat{\mathcal{W}}(\bar{z}))$ are shown in the following table. The bounds of  and  are already obtained in previous sections.

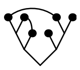
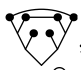


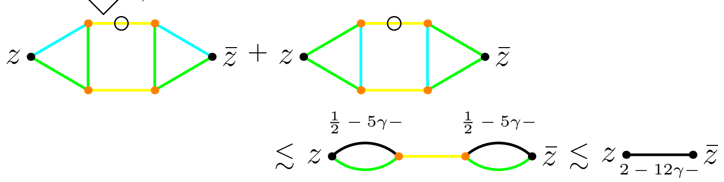

z $\bar{z}$
$1-10\gamma$
$1-10\gamma-$
$2-12\gamma$
$=$

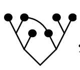
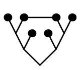
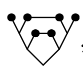


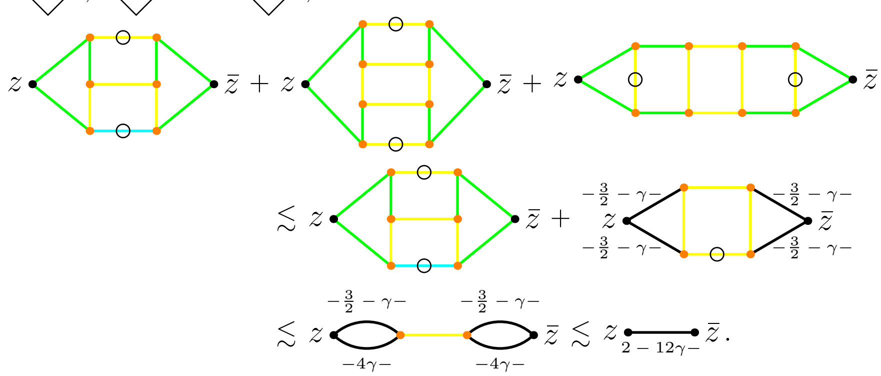
The estimates of remaining terms are obtained as follows. For ,

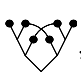


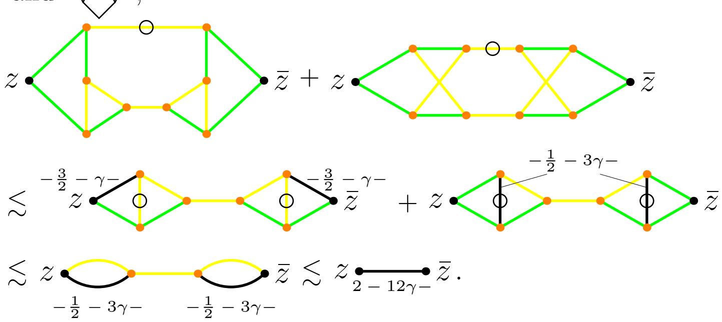
For  and ,



For ,  and ,

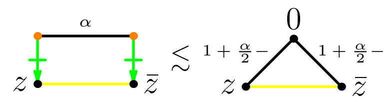


For  and ,



7.3. **Bounds of , , , and .** We classify all contractions into two types as in (5.1).

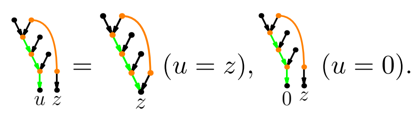
(1) The bounds of (1) are consequences of those of , , , and and obtained from Lemma 4.33 as follows.



Here $\alpha = -8\gamma$ (,), $-1-2\gamma$ ( ), $1-10\gamma-$ ( ), -4γ ( ).

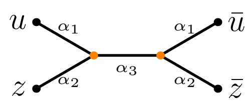
(2) We decompose graphs as in (5.2). The bounds of (2) are obtained directly. Since the bounds of and are already obtained in previous sections, we consider , , .

In all tables in the rest of this paper, a variable u represents 0 or z, and $\bar{u}$ represents 0 or $\bar{z}$, so that two trees can be treated at the same time. For example,



Our goal is to obtain graphs of the shape

(7.1)

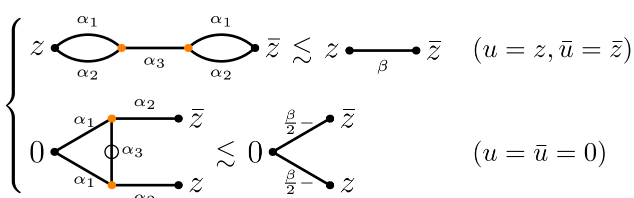


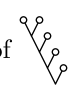
Let $\alpha_i \in (-3,0)$ $(i=1,2,3)$ and $\beta = 2\alpha_1 + 2\alpha_2 + \alpha_3 + 6$. Assume that

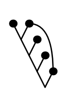
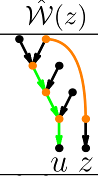
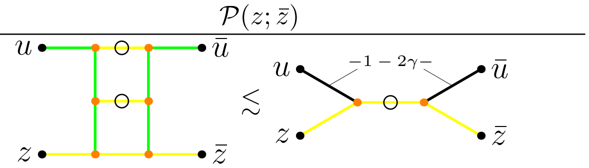
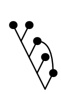
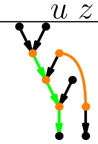
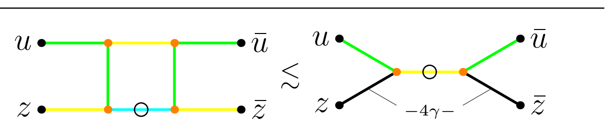
$$-3 < \alpha_1 + \alpha_2, \quad 2(\alpha_1 \vee \alpha_2) < \beta < 0. \tag{7.2}$$

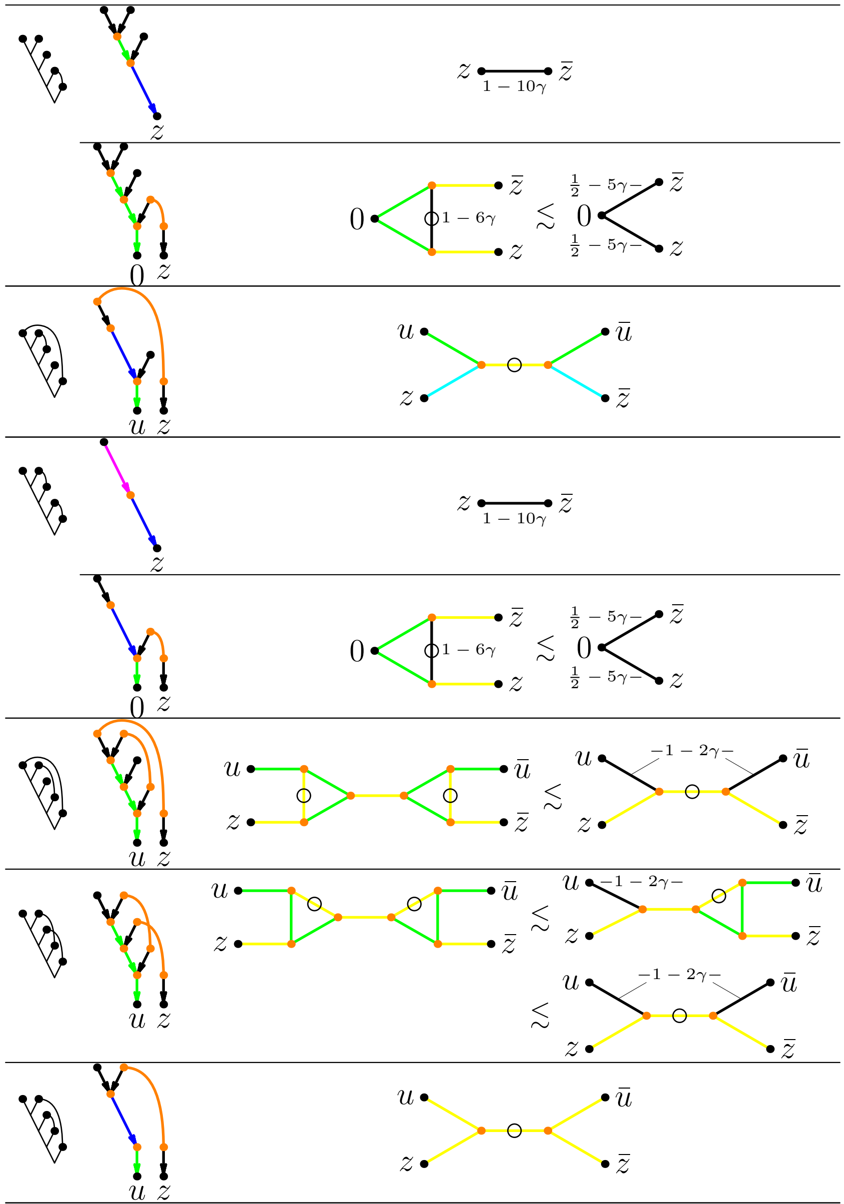
Then we have

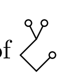
(7.1) =

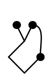
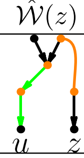
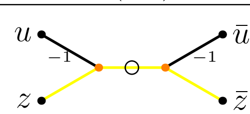


Bounds of 

	$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$
		
		

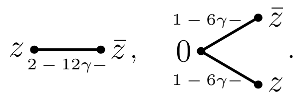


Bounds of 

	$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$
		

For , we obtain the bounds

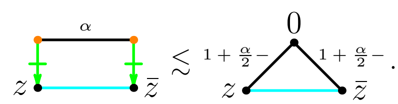
(7.3)



See Subsection 7.6.

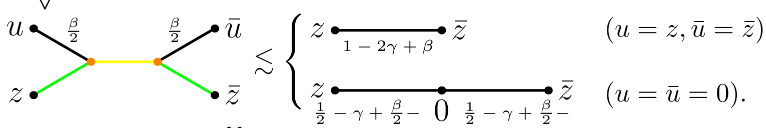
7.4. Bounds of , and . We classify all contractions into three types as in (6.1).

(1) The bounds of (1) are consequences of those of , and , and obtained from Lemma 4.33 as follows.



Here $\alpha = -8\gamma$ (,), $-1-2\gamma$ ().

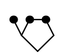
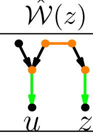
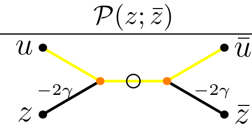
(2) We decompose graphs as in (6.2). The bound (2) are consequences of those of , and , and obtained as follows.



Here $\beta = 1-10\gamma-$ (,), -4γ ().

(3) We decompose graphs as in (6.4).

Bounds of

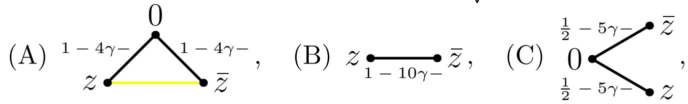
	$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$
		

For and , we obtain the bounds (7.3). See Subsection 7.6.

7.5. Bounds of , and . We classify all contractions into four types as in (6.5).

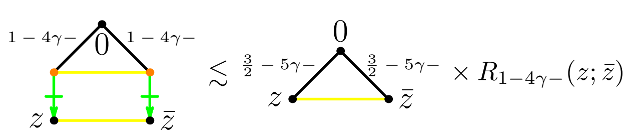
(1) Since the bounds of $\mathcal{P}(\tau)(z;\bar{z})$ ($\tau =$,) are any of

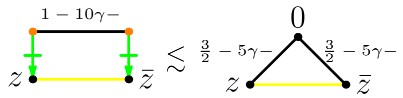
(7.4)

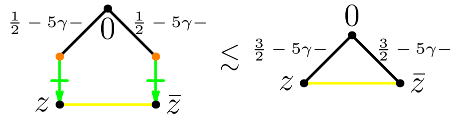


for and , we have the bounds of (1) as follows.

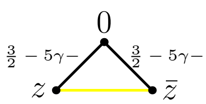
(A)



(B) 

(C) 

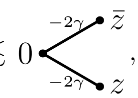
For the first one, similarly to (6.6) we have



$$\times R_{1-4\gamma-}(z;\bar{z}) \lesssim (\|z\|_{\mathfrak{s}}^{5-18\gamma-} + \|\bar{z}\|_{\mathfrak{s}}^{5-18\gamma-})\|z-\bar{z}\|_{\mathfrak{s}}^{-3+6\gamma}.$$

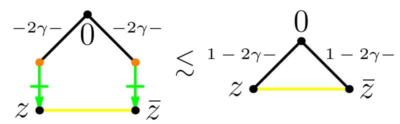
For , since

$$\mathcal{P}(\;)(z;\bar{z}) \lesssim$$

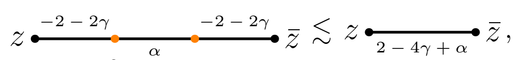


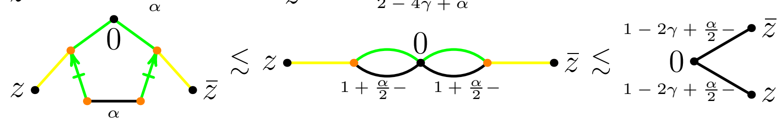
we have

$$\mathcal{P}(\;)(z;\bar{z}) \lesssim$$



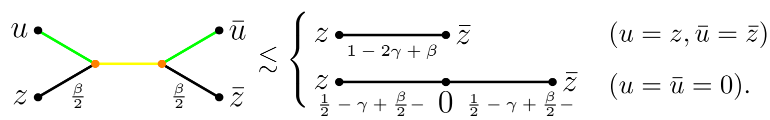
(2) We decompose graphs as in (6.7). By using the bounds of $\mathcal{P}(\tau)$ for $\tau =$, , , we have

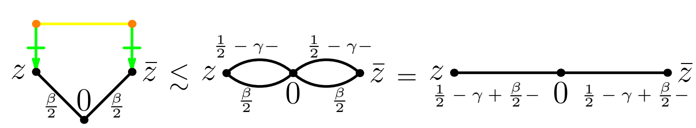
(a) 

(b) 

Here $\alpha = -8\gamma$ (,), $-1-2\gamma$ ( ).

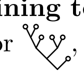
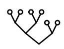
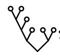
(3) We decompose graphs as in (6.8). The bounds of (3) are consequences of those of $\mathcal{P}(\tau)$ for $\tau =$, , similarly to (6.3) as follows.

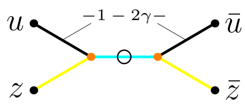
(a) (b) 

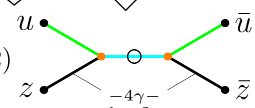
(c) 

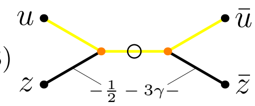
Here $\beta = 1-10\gamma-$ (,), -4γ ( ))

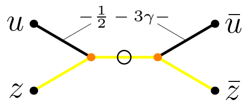
(4) We obtain the bounds (7.3). See Subsection 7.6. For , there are no contractions like (4).

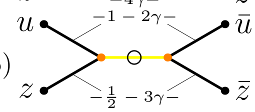
7.6. **Bounds of remaining terms.** The bounds of remaining terms are shown in the following tables. For ,  and , our goal is to obtain any of

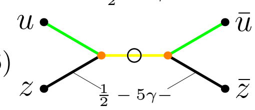
(A-1) 

(A-2) 

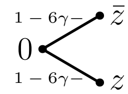
(A-3) 

(A-4) 

(A-5) 

(A-6) 

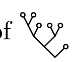
or

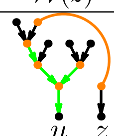
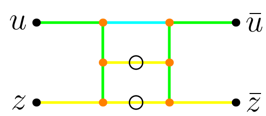
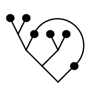
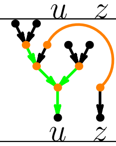
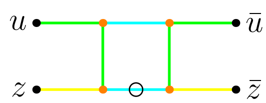
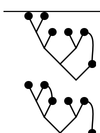
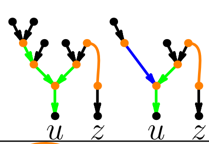
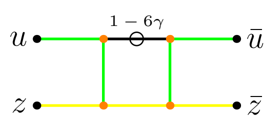
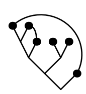
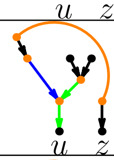
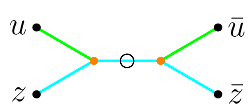
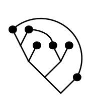
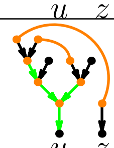
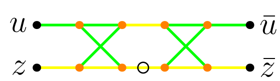
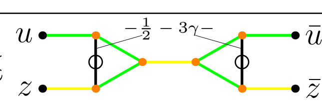
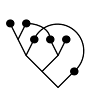
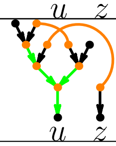
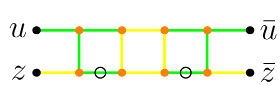
(B) 

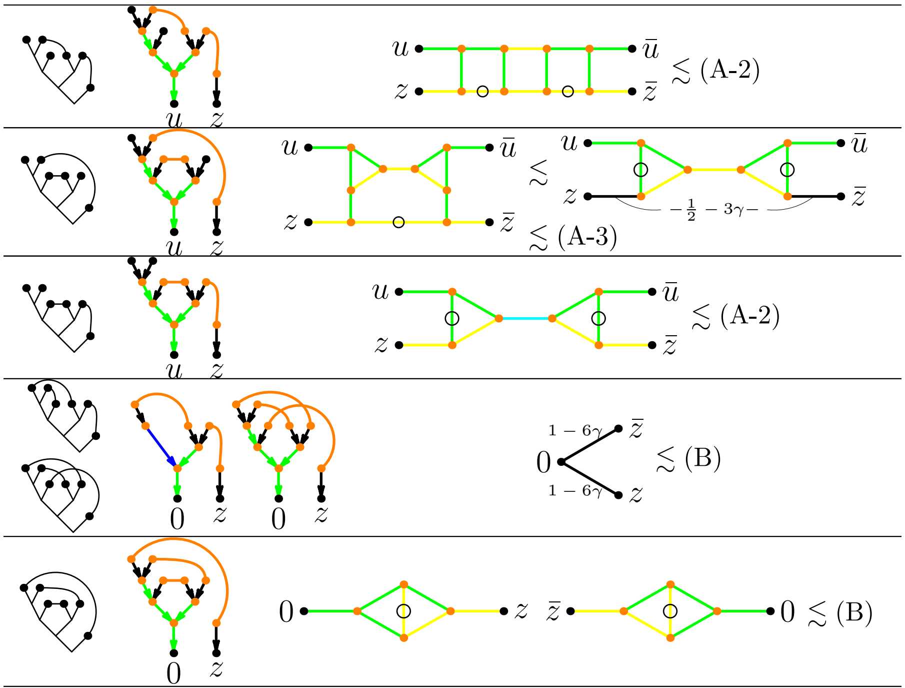
.

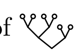
We can see that (A-1)-(A-6) satisfy (7.2) for $\beta = 2 - 12\gamma-$.

In the rest of this paper, we sometimes write two different points with the same label 0, in intermediate graphs, since each 0 comes from different kernels $\hat{\mathcal{W}}(z)$ and $\hat{\mathcal{W}}(\bar{z})$, respectively.

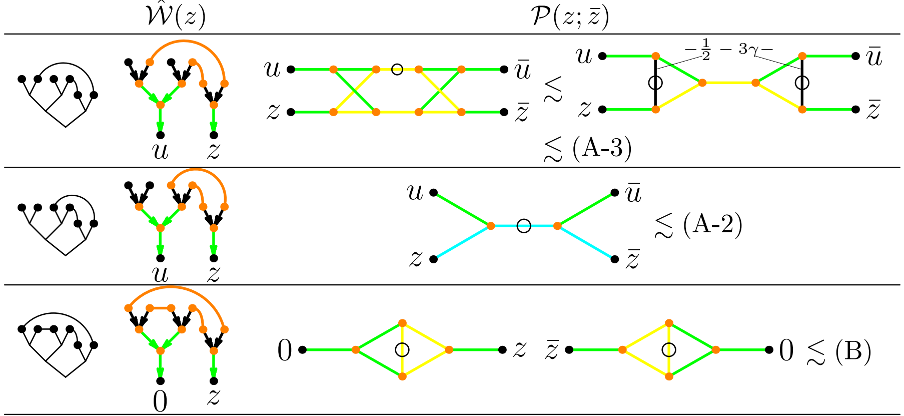
Bounds of 

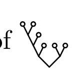
	$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$
		 $\lesssim$ (A-2)
		 $\lesssim$ (A-2)
		 $\lesssim$ (A-4)
		 $\lesssim$ (A-2)
		 $\lesssim$  $\lesssim$ (A-3)
		 $\lesssim$ (A-2)



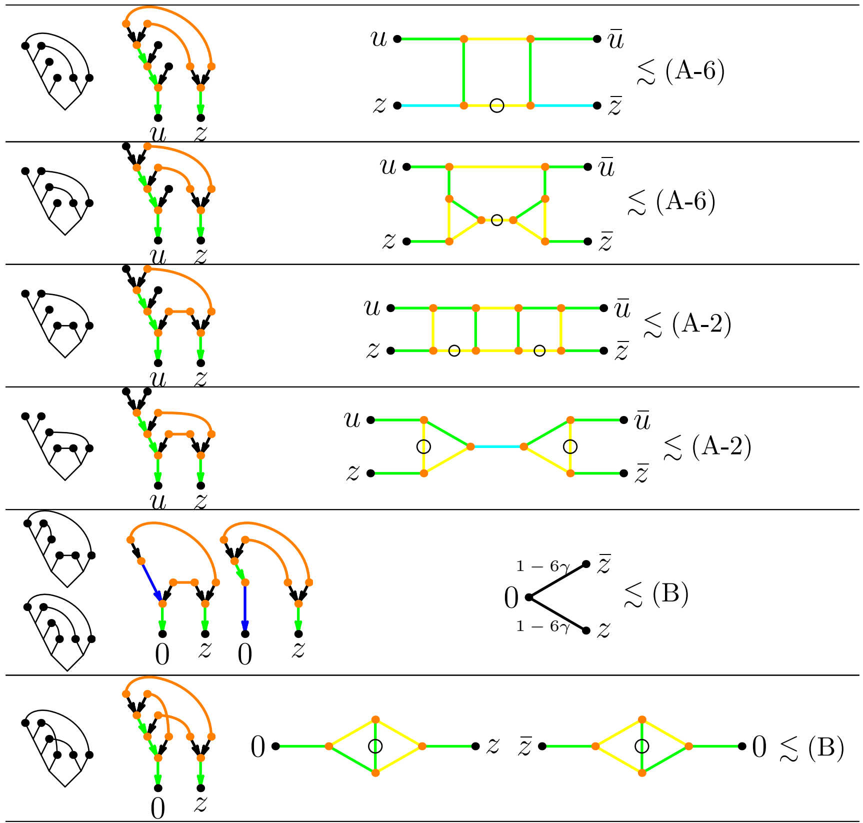
Bounds of 

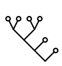
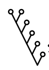
$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$

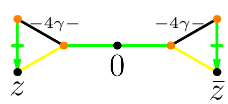


Bounds of 

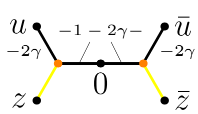
$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$

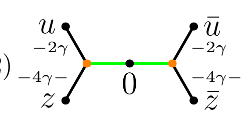


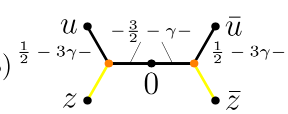
For  and , our goal is to obtain any of (A-1)-(A-6), (B),

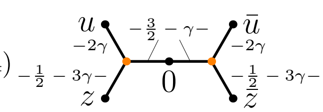
(C) 

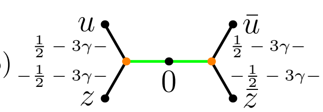
or

(D-1) 

(D-2) 

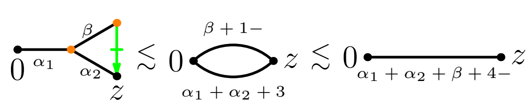
(D-3) 

(D-4) 

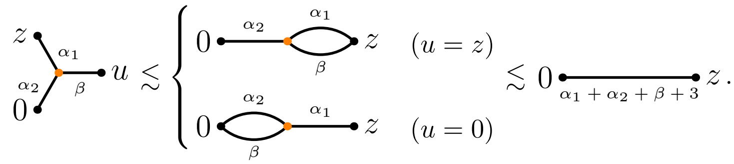
(D-5) 

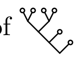
.

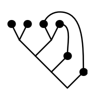
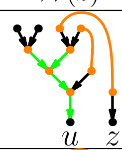
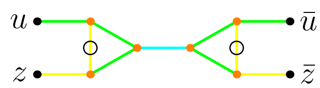
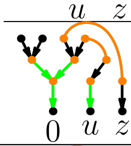
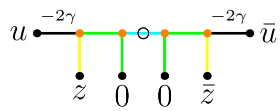
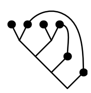
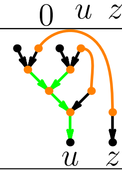
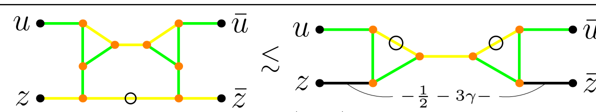
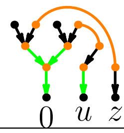
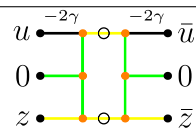
We obtain the required bounds from (C) and (D-1)-(D-5) as follows. Let $\alpha_1, \alpha_2 \in (-3, 0)$ and $\beta \in (-1, 0)$. Assume that $-3 < \alpha_1 + \alpha_2$. Then we have

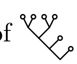
(7.5) 

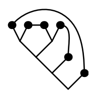
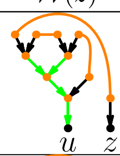
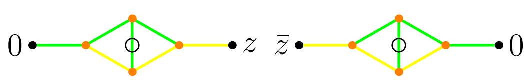
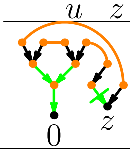
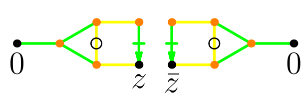
Let $\alpha_1, \alpha_2, \beta \in (-3, 0)$. Assume that $-3 < \alpha_i + \beta$ $(i = 1, 2)$ and $\alpha_1 + \alpha_2 + \beta + 3 < 0$. Then we have

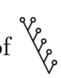
(7.6) 

Bounds of  (the case (6.9))

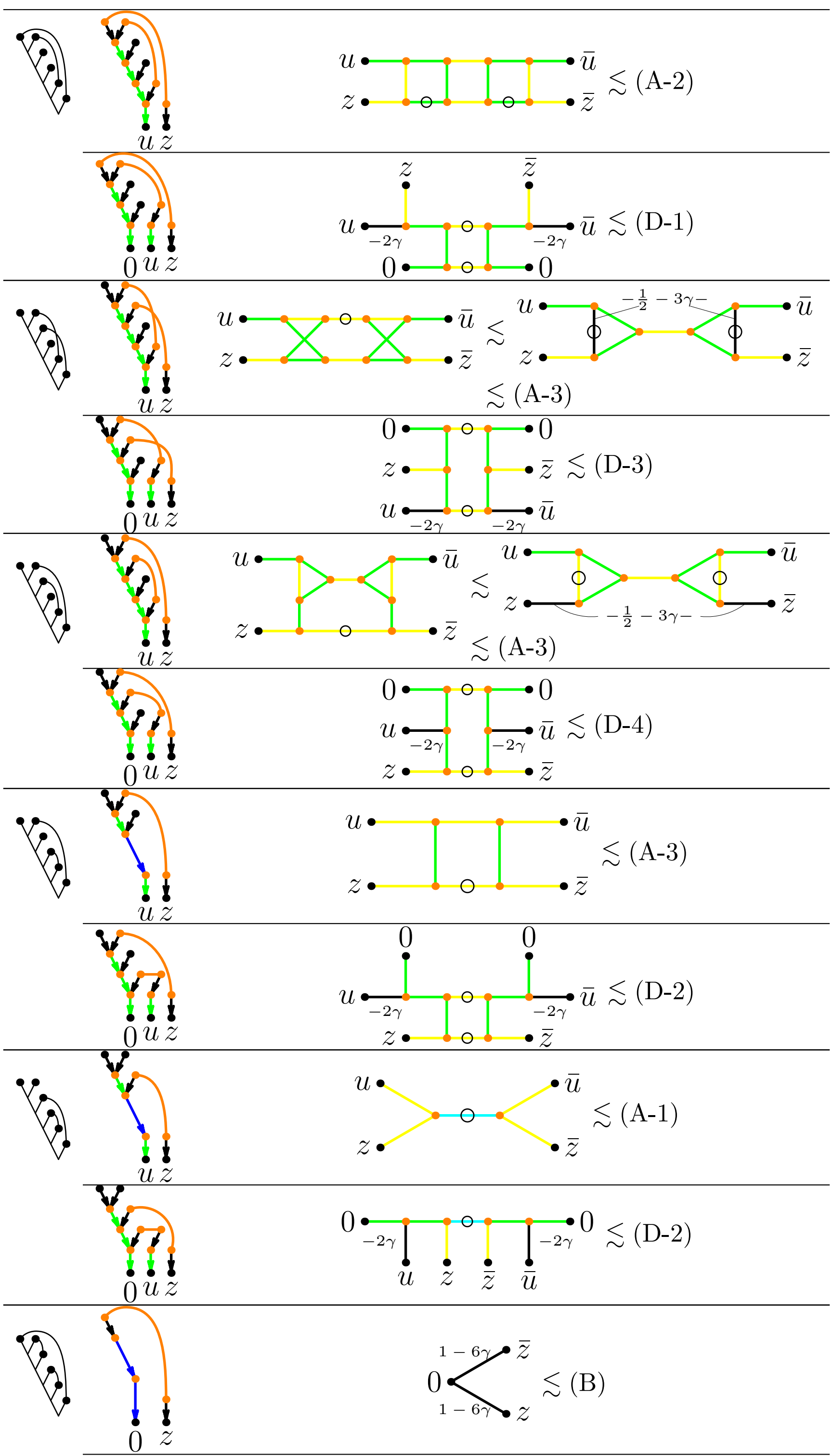
	$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$
		 $\lesssim$ (A-1)
		 $\lesssim$ (D-1)
		 $\lesssim$ (A-5)
		 $\lesssim$ (D-5)

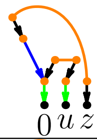
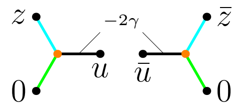
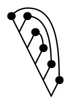
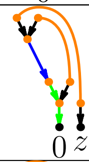
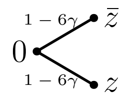
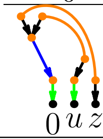
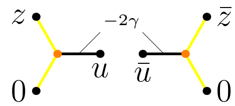
Bounds of  (the case (6.10))

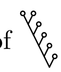
	$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$
		 $\lesssim$ (B)
		 $\lesssim$ (C)

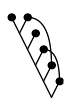
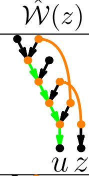
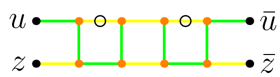
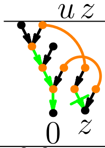
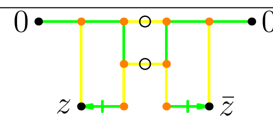
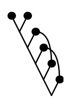
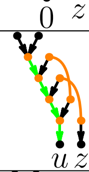
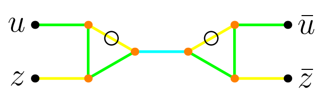
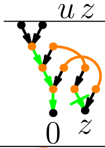
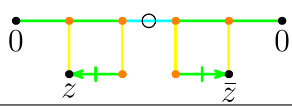
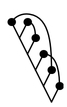
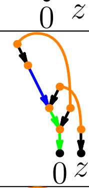
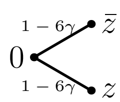
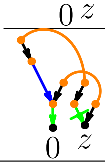
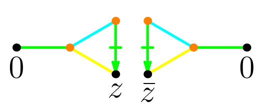
Bounds of  (the case (6.9))

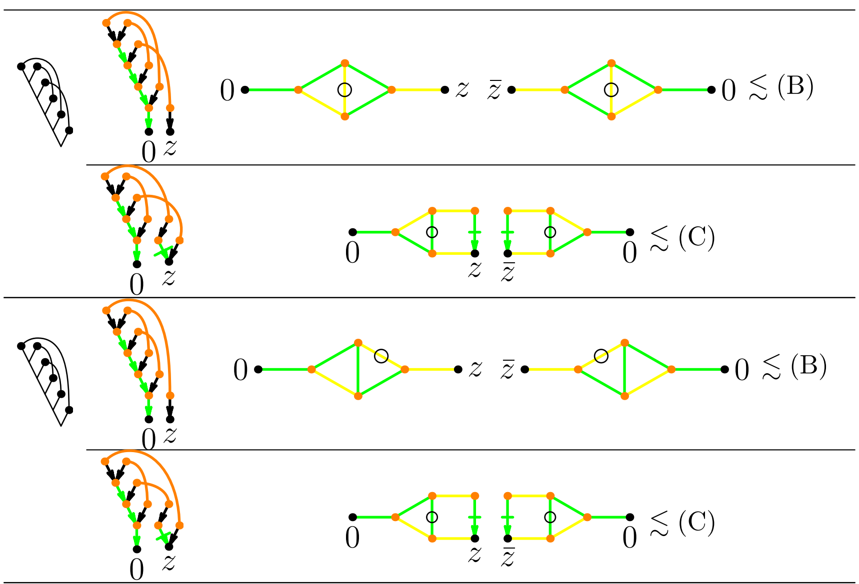
$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$


≲ (A-2)
≲ (D-1)
≲ (A-3)
≲ (D-3)
≲ (A-3)
≲ (D-4)
≲ (A-3)
≲ (D-2)
≲ (A-1)
≲ (D-2)
≲ (B)

			$\lesssim$ (D-2)
			$\lesssim$ (B)
			$\lesssim$ (D-1)

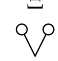
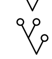
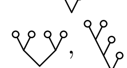
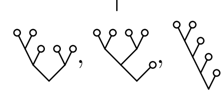
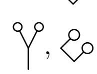
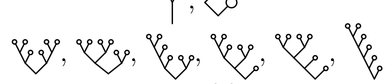
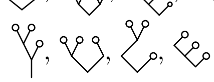
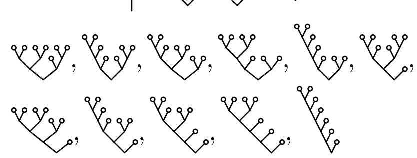
Bounds of  (the case (6.10))

	$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$	
			$\lesssim$ (A-1)
			$\lesssim$ (C)
			$\lesssim$ (A-1)
			$\lesssim$ (C)
			$\lesssim$ (B)
			$\lesssim$ (C)



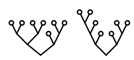
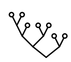
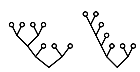
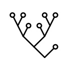
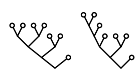
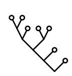
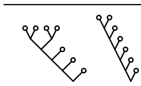
8. Renormalization in $\frac{3}{14} \leq \gamma < \frac{1}{4}$

Let $\frac{3}{14} \leq \gamma < \frac{1}{4}$. Then all elements in $\mathcal{F}_-$ are as follows.

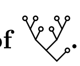
Homogeneity	Symbol
$-\frac{3}{2} - \gamma - \kappa$	Ξ
$-1 - 2\gamma - 2\kappa$	
$-\frac{1}{2} - 3\gamma - 3\kappa$	
$-4\gamma - 4\kappa$	
$-\frac{1}{2} - \gamma - \kappa$	
$\frac{1}{2} - 5\gamma - 5\kappa$	
$-2\gamma - 2\kappa$	
$1 - 6\gamma - 6\kappa$	
$\frac{1}{2} - 3\gamma - 3\kappa$	
$\frac{3}{2} - 7\gamma - 7\kappa$	
0	$\mathbf{1}$

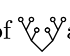
It remains to obtain the bounds for elements with homogeneity $\frac{3}{2}-7\gamma-7\kappa$. The other bounds are same as in Section 7.

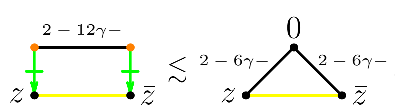
Proposition 8.1. *For every τ with $\|\tau\| = \frac{3}{2} - 7\gamma - 7\kappa$, we have the bounds of $\mathcal{P}(z;\bar{z})$ as follows.*

Symbol	$\mathcal{P}(z;\bar{z})$
	$\|z\|_{\mathfrak{s}}^{1-4\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{1-4\gamma-\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{1-6\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\quad \|z\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$
	$\|z\|_{\mathfrak{s}}^{\frac{3}{2}-5\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{3}{2}-5\gamma-\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-4\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\quad \|z\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$
	$(\|z\|_{\mathfrak{s}}^{5-18\gamma-\theta}+\|\bar{z}\|_{\mathfrak{s}}^{5-18\gamma-\theta})\|z-\bar{z}\|_{\mathfrak{s}}^{-2+4\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\quad \|z\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$
	$\|z\|_{\mathfrak{s}}^{2-6\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{2-6\gamma-\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\quad \|z\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$
	$(\|z\|_{\mathfrak{s}}^{6-20\gamma-\theta}+\|\bar{z}\|_{\mathfrak{s}}^{6-20\gamma-\theta})\|z-\bar{z}\|_{\mathfrak{s}}^{-3+6\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\quad \|z\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$
	$(\|z\|_{\mathfrak{s}}^{7-22\gamma-\theta}+\|\bar{z}\|_{\mathfrak{s}}^{7-22\gamma-\theta})\|z-\bar{z}\|_{\mathfrak{s}}^{-4+8\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\quad \|z\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$
	$(\|z\|_{\mathfrak{s}}^{9-30\gamma-\theta}+\|\bar{z}\|_{\mathfrak{s}}^{9-30\gamma-\theta})\|z-\bar{z}\|_{\mathfrak{s}}^{-6+16\gamma}$, $\|z-\bar{z}\|_{\mathfrak{s}}^{3-14\gamma-\theta}$, $\quad \|z\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}\|\bar{z}\|_{\mathfrak{s}}^{\frac{3}{2}-7\gamma-\theta}$

Here θ is any sufficiently small number. The bounds of $\mathcal{P}^{(\epsilon)}(z;\bar{z})$ are obtained by multiplying ϵ^{θ} to above bounds whose indexes are slightly subtracted as in Proposition 5.1.

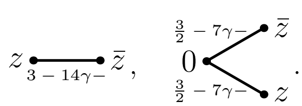
8.1. **Bounds of** **.** We classify all contractions into two types as in (5.1).

(1) The bounds of (1) are consequences of those of  and obtained from Lemma 4.33 as follows.

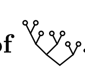


(2) We decompose graphs as in (5.2). We obtain the bounds

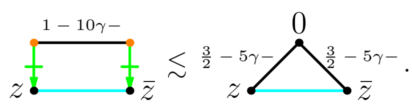
(8.1)



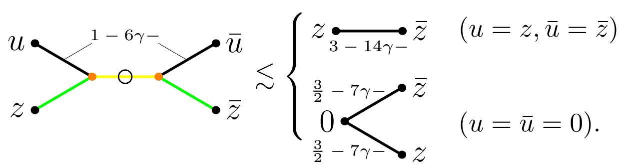
See Subsection 8.10.

8.2. **Bounds of** **.** We classify all contractions into three types as in (6.1).

(1) The bounds of (1) are consequence of those of and obtained by Lemma 4.33 as follows.



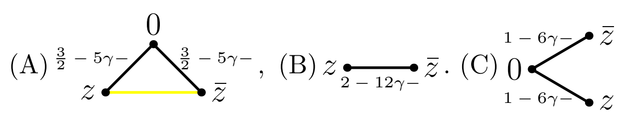
(2) We decompose graphs as in (6.2). The bounds are consequences of those of and obtained as follows.



(3) We decompose graphs as in (6.4). We obtain the bounds (8.1). See Subsection 8.10.

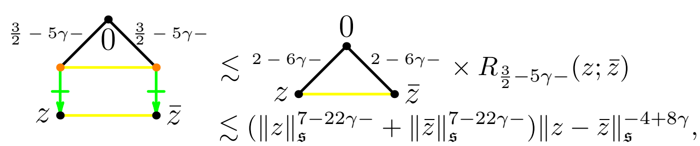
8.3. **Bounds of .** We classify all contractions into four types as in (6.5).

(1) Since the bounds of $\mathcal{P}(\ )(z;\bar{z})$ are any of

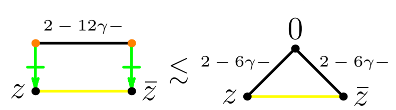


we have the bounds of (1) as follows.

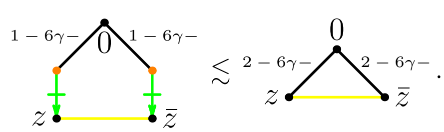
(A)



(B)

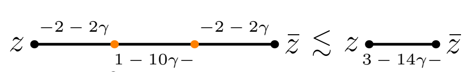


(C)

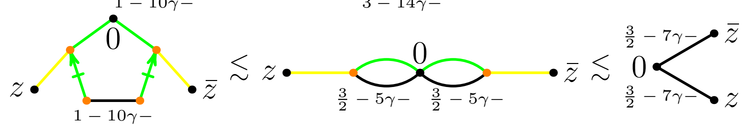


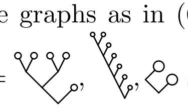
(2) We decompose graphs as in (6.7). By using the bounds of $\mathcal{P}(\ )$, we have

(a)

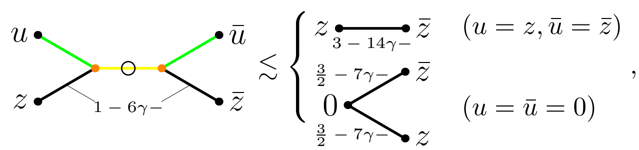


(b)

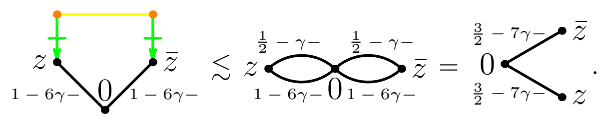


(3) We decompose graphs as in (6.8). The bounds of (3) are consequences of those of $\mathcal{P}(\tau)$ for $\tau =$  similarly to (6.3) as follows.

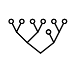
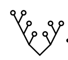
(a) (b)

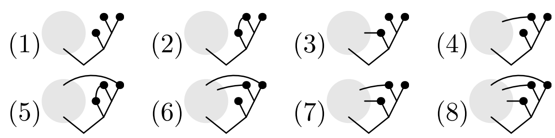


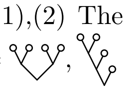
(c)

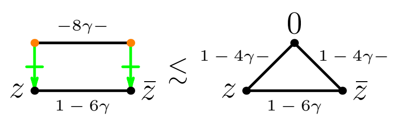


(4) We obtain the bounds (8.1). See Subsection 8.10.

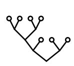
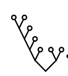
8.4. **Bounds of**  **and** **.** We classify all contractions into eight types as follows.

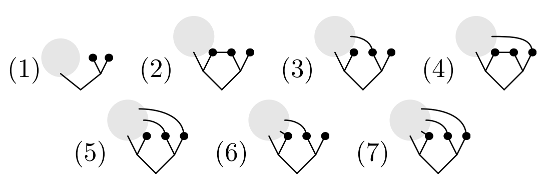


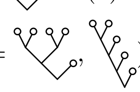
(1),(2) The bounds of (1) and (2) are consequences of those of $\mathcal{P}(\tau)(z;\bar{z})$ for $\tau =$  and obtained from Lemma 4.33 as follows.



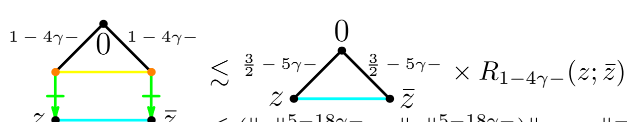
(3)-(7) We obtain the bounds (8.1). See Subsection 8.8.
(8) We obtain the bounds (8.1). See Subsection 8.10.

8.5. **Bounds of**  **and** **.** We classify all contractions into seven types as follows.



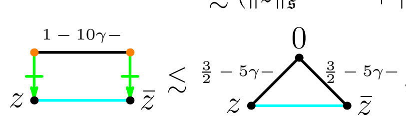
(1) From the bounds of $\mathcal{P}(\tau)(z;\bar{z})$ ($\tau =$ ) in (7.4), we have the bounds of (1) as follows.

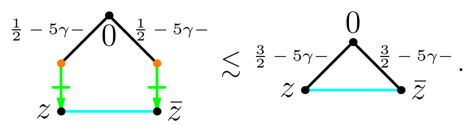
(A)



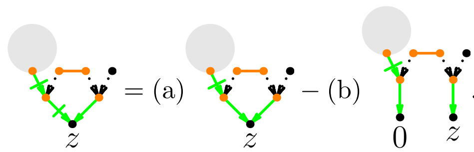
$$\lesssim (\|z\|_{\mathfrak{s}}^{5-18\gamma-} + \|\bar{z}\|_{\mathfrak{s}}^{5-18\gamma-})\|z-\bar{z}\|_{\mathfrak{s}}^{-2+4\gamma},$$

(B)

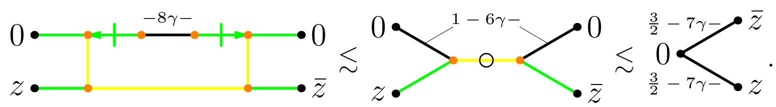


(C) 

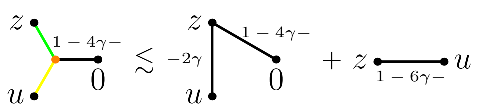
(2) We decompose graphs as follows.



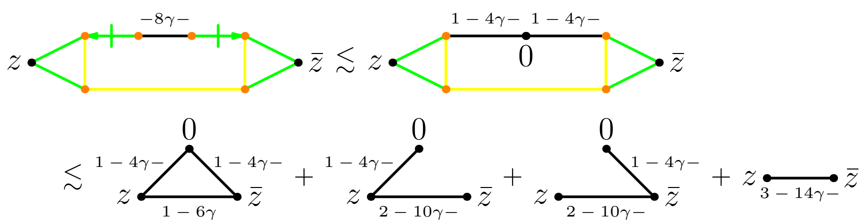
For (b),



For (a), we use



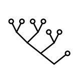
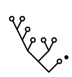
which is proved from $\|0-\bullet\|_{\mathfrak{s}}^{1-4\gamma-} \leq \|z\|_{\mathfrak{s}}^{1-4\gamma-} + \|z-\bullet\|_{\mathfrak{s}}^{1-4\gamma-}$. Then we have

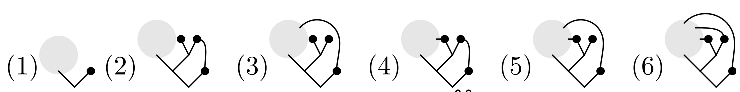


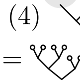
$$\lesssim (\|z\|_{\mathfrak{s}}^{2-8\gamma-} + \|\bar{z}\|_{\mathfrak{s}}^{2-8\gamma-})\|z-\bar{z}\|_{\mathfrak{s}}^{1-6\gamma}.$$

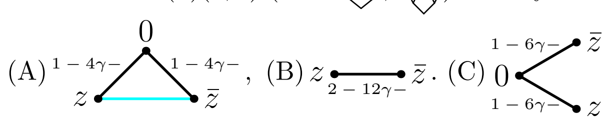
(3)-(6) We obtain the bounds (8.1). See Subsection 8.8.

(7) We obtain the bounds (8.1). See Subsection 8.10.

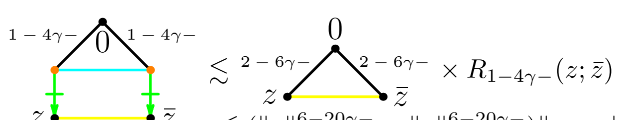
8.6. **Bounds of**  **and** **.** We classify all contractions into six types as follows.



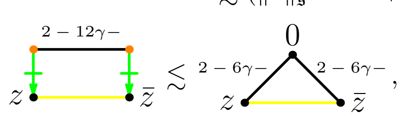
(1) Since the bounds of $\mathcal{P}(\tau)(z;\bar{z})$ ($\tau =$ , ) are any of

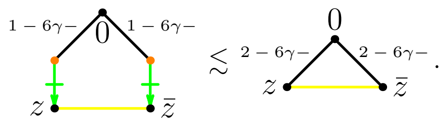


we have the bounds of (1) as follows.

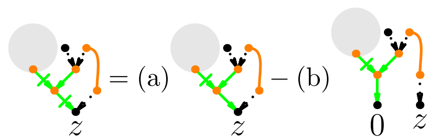
(A) 

$$\lesssim (\|z\|_{\mathfrak{s}}^{6-20\gamma-} + \|\bar{z}\|_{\mathfrak{s}}^{6-20\gamma-})\|z-\bar{z}\|_{\mathfrak{s}}^{-3+6\gamma},$$

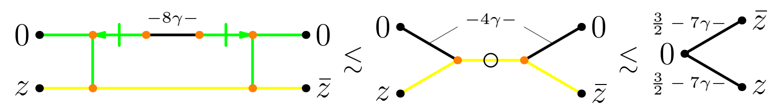
(B) 

(C) 

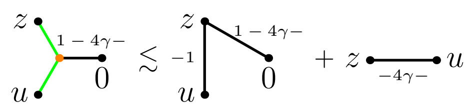
(2) We decompose graphs as follows.



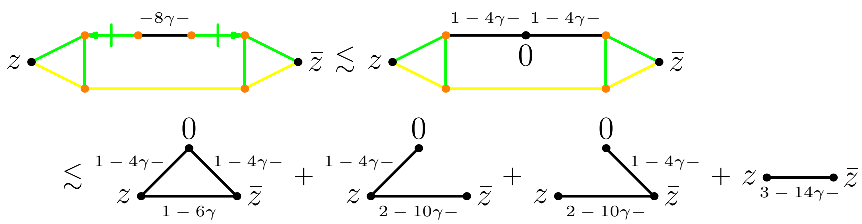
For (b),



For (a), we use



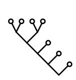
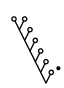
which is proved from $\|0-\bullet\|_{\mathfrak{s}}^{1-4\gamma-}\le\|z\|_{\mathfrak{s}}^{1-4\gamma-}+\|z-\bullet\|_{\mathfrak{s}}^{1-4\gamma-}$. Then we have

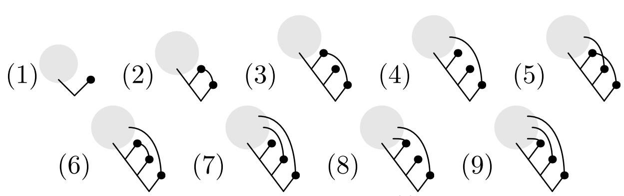


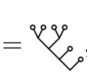
$$\lesssim(\|z\|_{\mathfrak{s}}^{2-8\gamma-}+\|\bar z\|_{\mathfrak{s}}^{2-8\gamma-})\|z-\bar z\|_{\mathfrak{s}}^{1-6\gamma}.$$

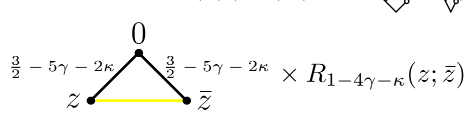
(3)-(5) We obtain the bounds (8.1). See Subsection 8.8.

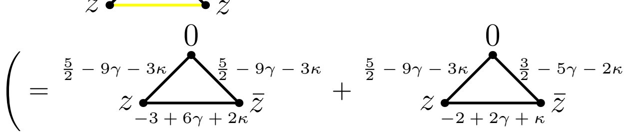
(6) We obtain the bounds (8.1). See Subsection 8.10.

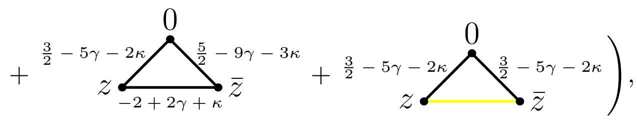
8.7. **Bounds of**  **and** **.** We classify all contractions into nine types as follows.



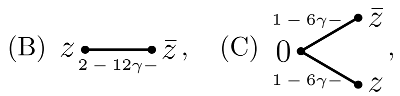
(1) Since the bounds of $\mathcal{P}(\tau)(z;\bar z)$ ($\tau=$ , ) are any of

(A) 

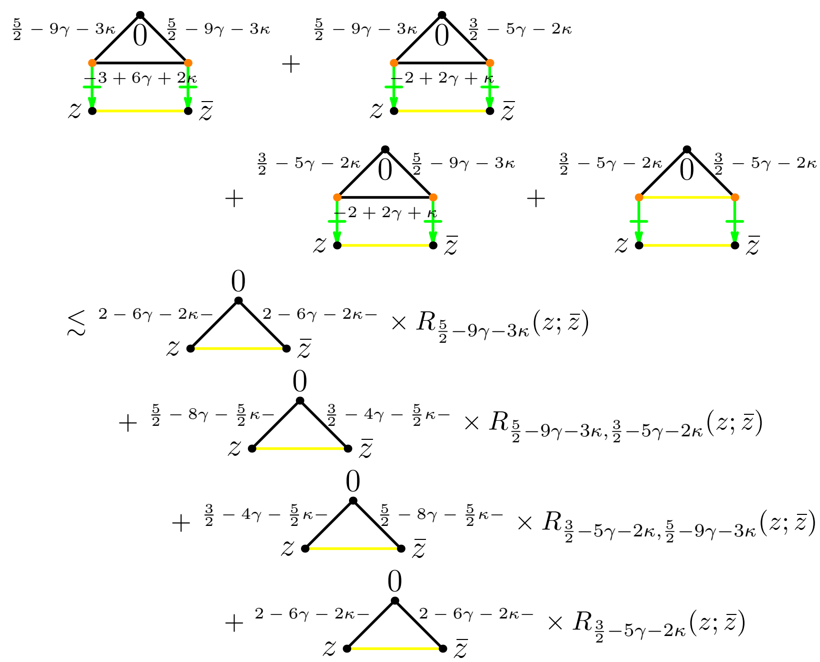




$\kappa > 0$ is any sufficiently small number

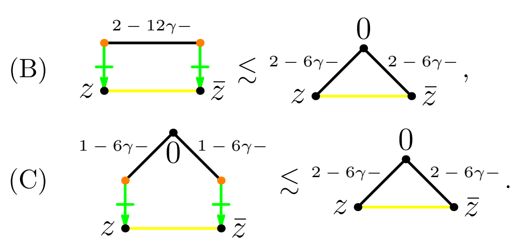


for (A) we have

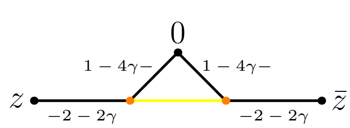


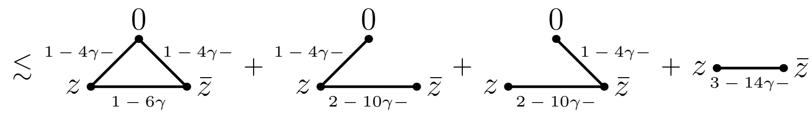
$$\lesssim (\|z\|_{\mathfrak{s}}^{9-30\gamma-10\kappa-} + \|\bar{z}\|_{\mathfrak{s}}^{9-30\gamma-10\kappa-})\|z-\bar{z}\|_{\mathfrak{s}}^{-6+16\gamma+6\kappa}.$$

For (B) and (C) we have



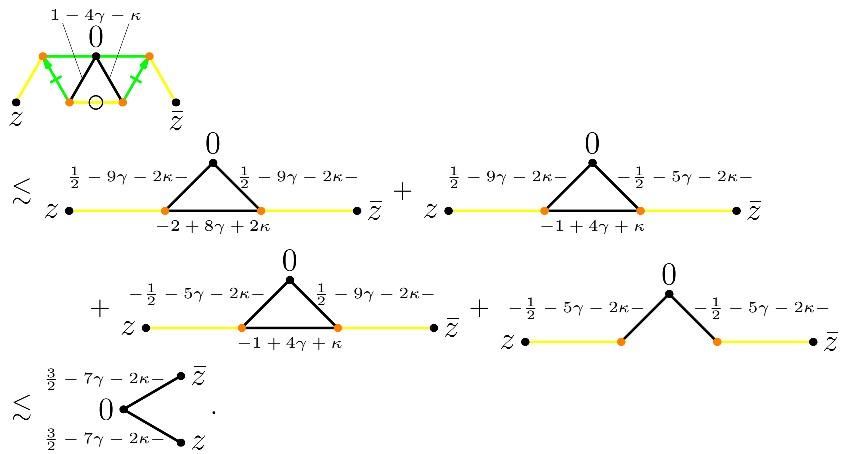
(2) We decompose graphs as in (6.7). We apply estimates (A)-(C) in (7.4) to the graphs (a) and (b). By applying (A) to (a), we have



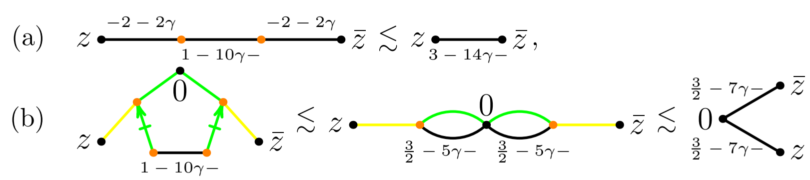


$$\lesssim (\|z\|_{\mathfrak{s}}^{2-8\gamma-} + \|\bar{z}\|_{\mathfrak{s}}^{2-8\gamma-})\|z-\bar{z}\|_{\mathfrak{s}}^{1-6\gamma},$$

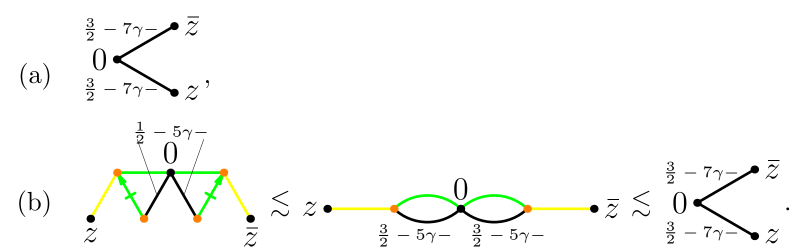
which is proved by a similar way to the proof of Lemma 4.34. By applying (A) to (b), we have



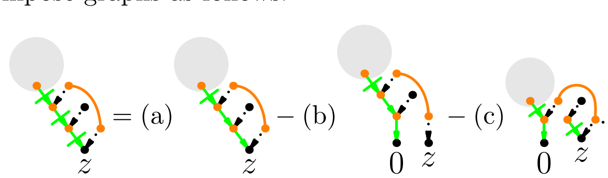
Here, $\kappa > 0$ is a sufficiently small number. By applying (B) to (a) and (b), we have



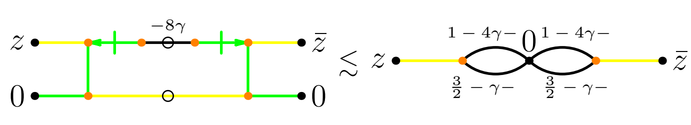
By applying (C) to (a) and (b), we have

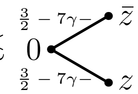


(3) We decompose graphs as follows.

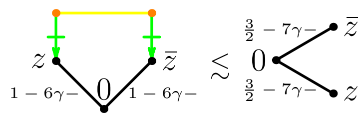


For (b),

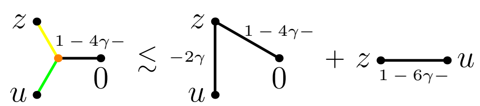


$\lesssim$ 

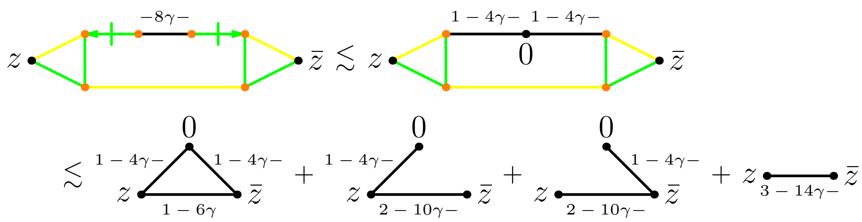
For (c),



For (a), we use

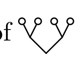
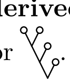
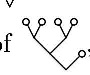
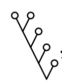
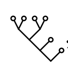
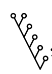
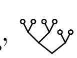
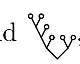


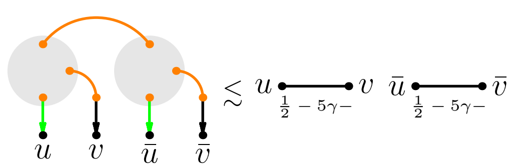
which is proved from $\|0 - \bullet\|_{\mathfrak{s}}^{1-4\gamma-} \leq \|z\|_{\mathfrak{s}}^{1-4\gamma-} + \|z - \bullet\|_{\mathfrak{s}}^{1-4\gamma-}$. Then we have

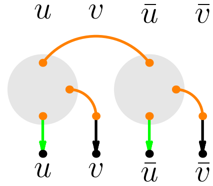


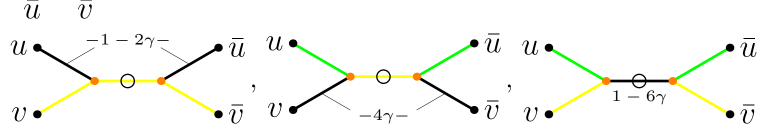
$\lesssim (\|z\|_{\mathfrak{s}}^{2-8\gamma-} + \|\bar{z}\|_{\mathfrak{s}}^{2-8\gamma-})\|z - \bar{z}\|_{\mathfrak{s}}^{1-6\gamma}.$

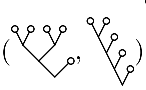
(4)-(8) We obtain the bounds (8.1). See Subsection 8.8.
(9) We obtain the bounds (8.1). See Subsection 8.10.

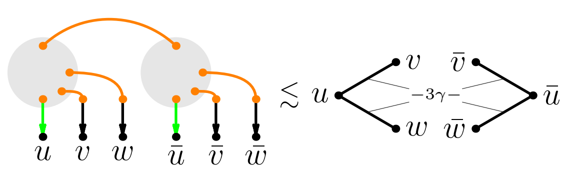
8.8. **Bounds of remaining terms derived from already known estimates.** Let  be every contractions of  or . Then we have the following estimates. They are already obtained in estimates of , , , ,  and , except (c).

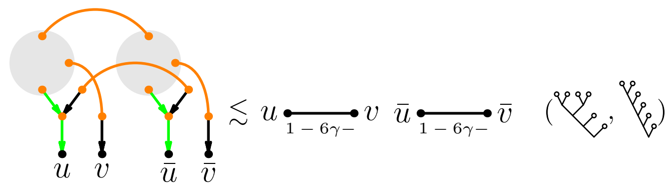
(a) 

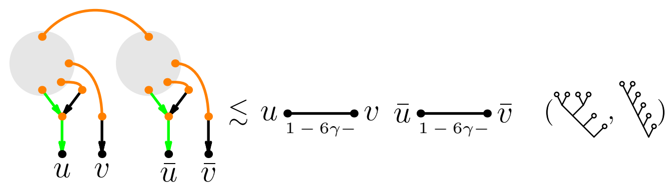
(b) 

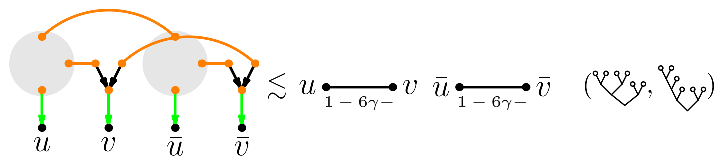
$\lesssim$ any of 

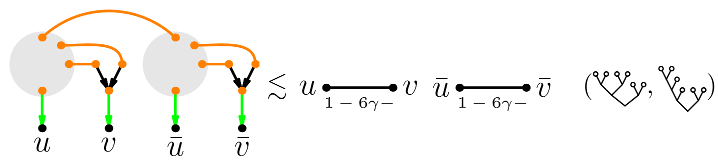
()

(c)  (See Subsection 8.9)

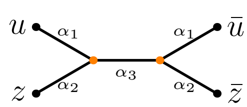
(d) 

(e) 

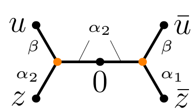
(f) 

(g) 

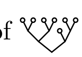
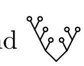
We prove the bounds of remaining terms by using above estimates. For graphs containing u and $\bar{u}$, our goal is to obtain

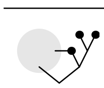
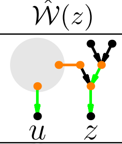
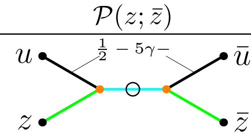
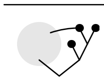
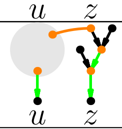
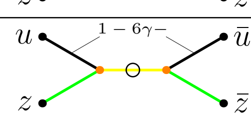


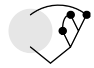
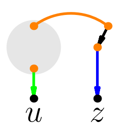
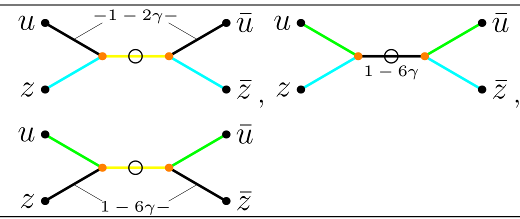
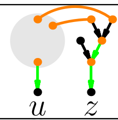
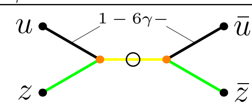
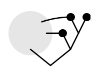
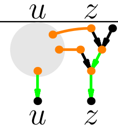
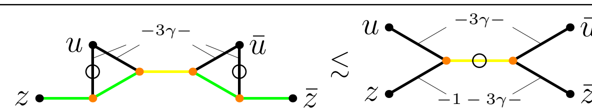
$$\begin{cases} \alpha_i \in (-3,0)\ (i=1,2,3),\ \alpha_1+\alpha_2 \in (-3,0), \\ 2\alpha_1+2\alpha_2+\alpha_3+6 < 3-14\gamma, \\ \alpha_1,\alpha_2 < \frac{3}{2}-7\gamma \end{cases}$$



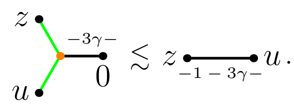
$$\begin{cases} \alpha_i \in (-3,0)\ (i=1,2),\ \beta \in (-3,0), \\ -3 < \alpha_i+\beta < 0\ (i=1,2), \\ \alpha_1+\alpha_2+\beta+3 < \frac{3}{2}-7\gamma \end{cases}.$$

Bounds of  and 

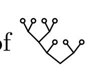
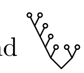
	$\hat{\mathcal{W}}(z)$	Use	$\mathcal{P}(z;\bar{z})$
		(a)	
		(f)	

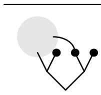
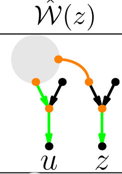
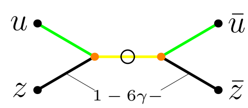
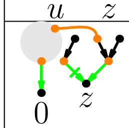
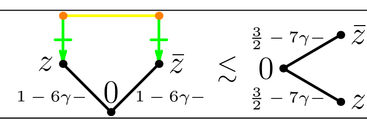
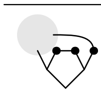
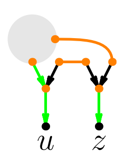
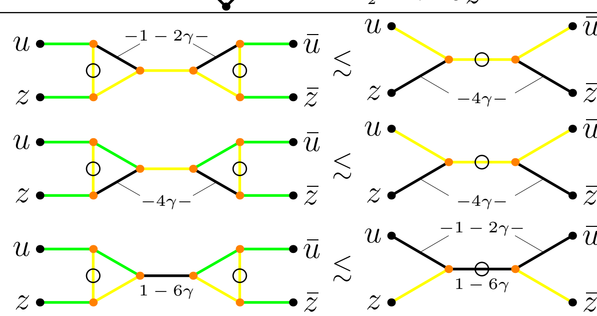
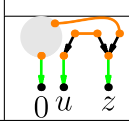
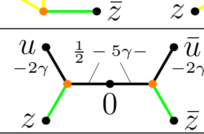
		(b)	
		(g)	
		(c)	

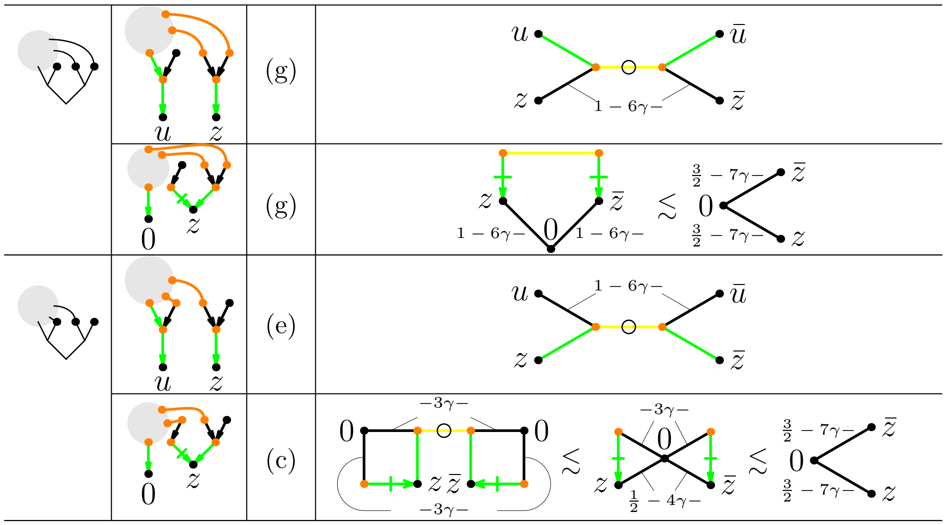
In the last one, we use

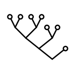
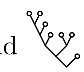


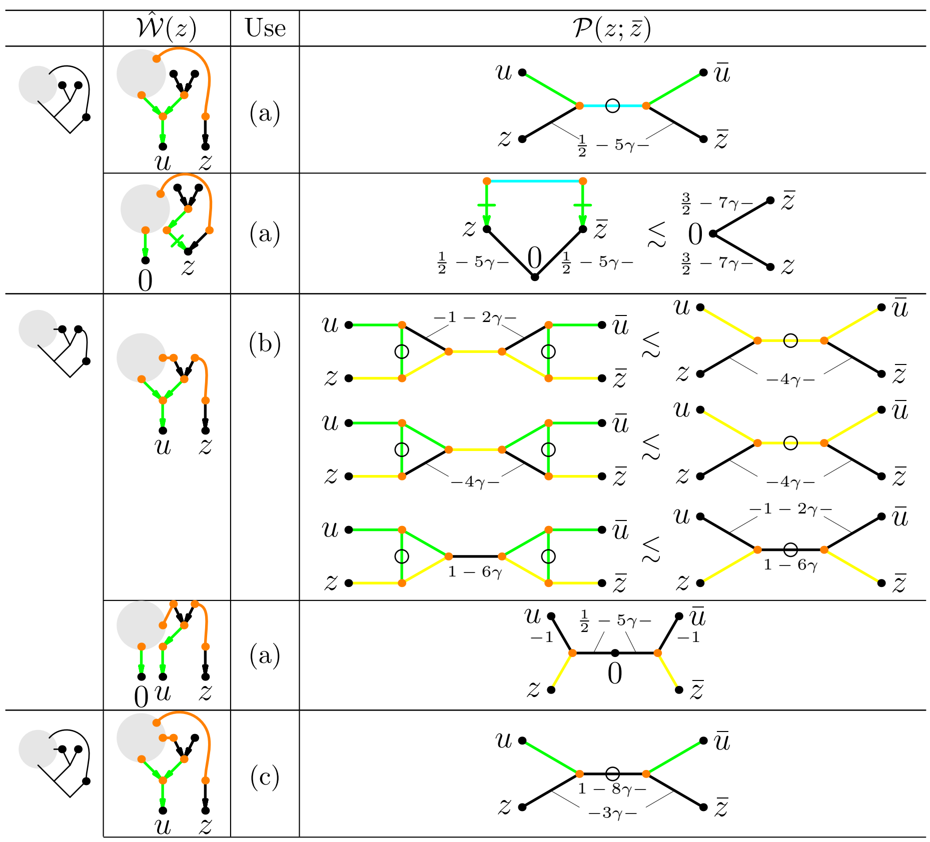
It is a "one side" version of Lemma 4.31, and proved by a similar way.

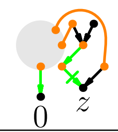
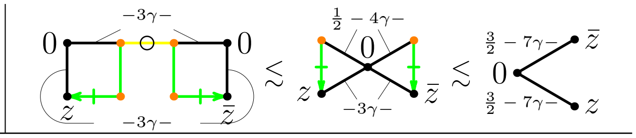
Bounds of  and 

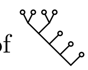
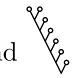
	$\hat{\mathcal{W}}(z)$	Use	$\mathcal{P}(z;\bar{z})$
		(f)	
		(f)	
		(b)	
		(a)	

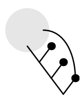
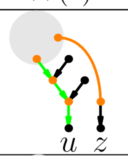
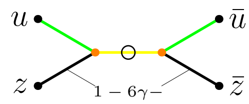
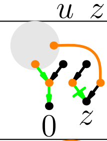
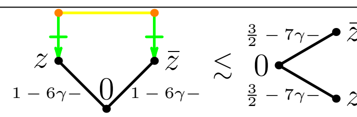
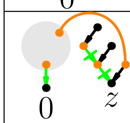
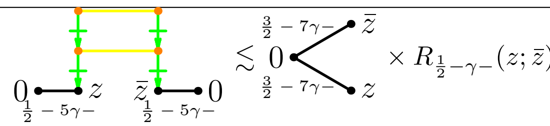
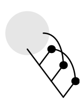
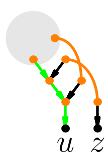
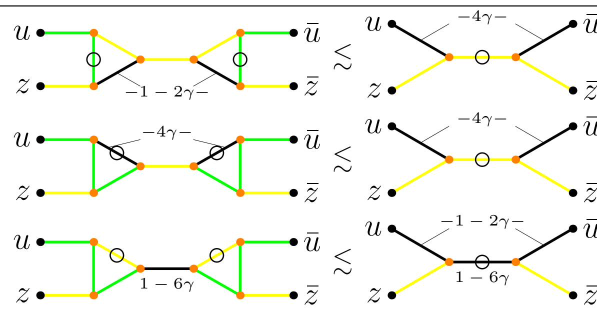
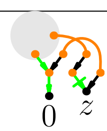
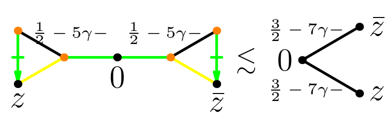
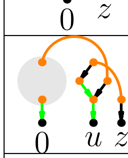
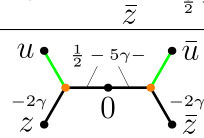
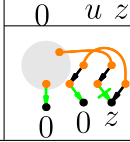
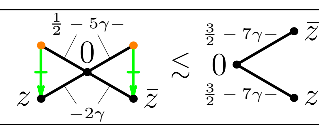


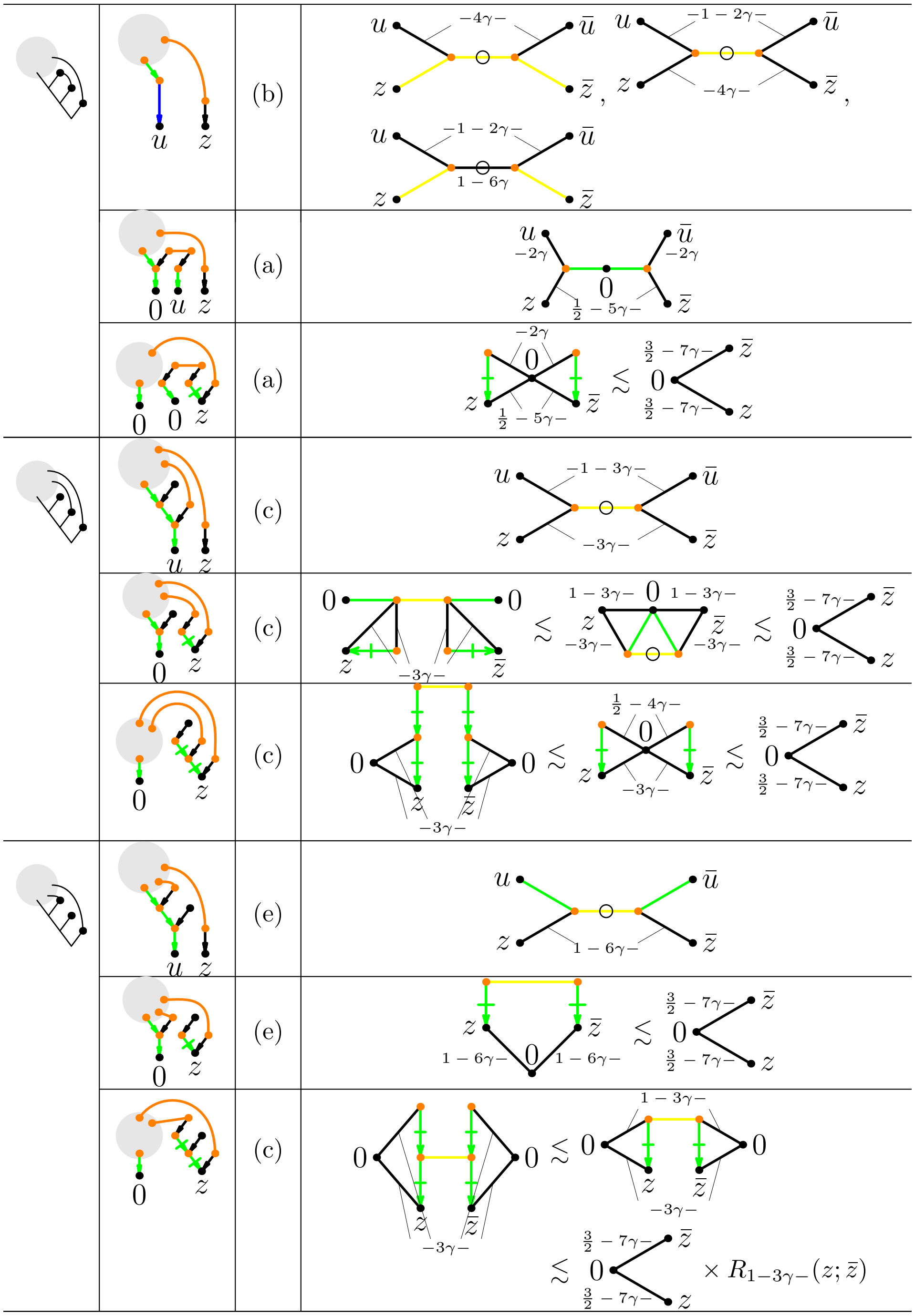
Bounds of  and 



		(c)	

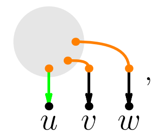
Bounds of  and 

	$\hat{\mathcal{W}}(z)$	Use	$\mathcal{P}(z;\bar z)$
		(d)	
		(d)	
		(a)	
		(b)	
		(a)	
		(a)	
		(a)	

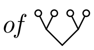
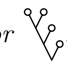


8.9. **Proof of estimates (c).** We prove the fact used in Subsection 8.8.

Lemma 8.2. *Let $N \in \mathbb{N}$ and $\mathcal{W}(u, v, w; \{w_i\}_{i=1,\dots,N})$ be a function on $(\mathbb{R}^2)^3 \times (\mathbb{R} \times \mathbb{T})^N$ written as follows.*

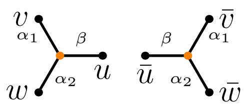


,

where *is any of graphs which are given by contractions of*  *or* *. Let $C > 0$ be any sufficiently small number such that all variables in the above graph are in $B_{\mathfrak{s}}(0, \frac{1}{4})$ as long as $u, v, w \in B_{\mathfrak{s}}(0, C)$. Then for every $\kappa > 0$ and $u, v, w, \bar{u}, \bar{v}, \bar{w} \in B_{\mathfrak{s}}(0, C)$, we have*

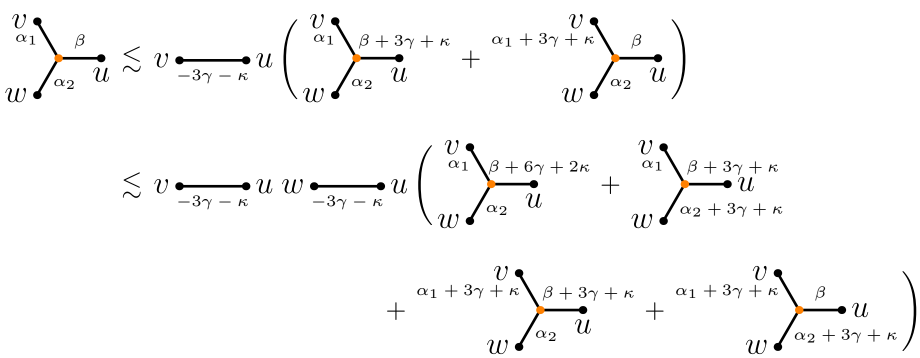
$$\left|(\mathcal{W}(u, v, w; \cdot), \mathcal{W}(\bar{u}, \bar{v}, \bar{w}; \cdot))_{L^2((\mathbb{R}\times\mathbb{T})^N)}\right| \lesssim \|u - v\|_{\mathfrak{s}}^{-3\gamma-\kappa} \|u - w\|_{\mathfrak{s}}^{-3\gamma-\kappa} \times \|\bar{u} - \bar{v}\|_{\mathfrak{s}}^{-3\gamma-\kappa} \|\bar{u} - \bar{w}\|_{\mathfrak{s}}^{-3\gamma-\kappa}.$$

proof. We prove that all graphs $\left|(\mathcal{W}(u, v, w; \cdot), \mathcal{W}(\bar{u}, \bar{v}, \bar{w}; \cdot))_{L^2((\mathbb{R}\times\mathbb{T})^N)}\right|$ are contracted into the following form.

(8.2) 

$$\begin{cases} -3 < \alpha_i < -3\gamma \ (i = 1, 2), \ -3 < \beta < -6\gamma \\ -3 < \alpha_1 + \alpha_2, \\ \alpha_1 + \alpha_2 + \beta + 3 + 6\gamma < 0 \ \text{(sufficiently close).} \end{cases}$$

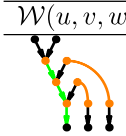
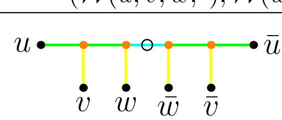
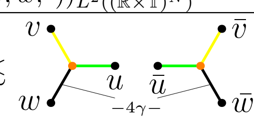
We can derive the required bound from this graph by a similar way to Lemma 4.31. By using (4.7) repeatedly, for $\kappa > 0$ such that $\beta + 6\gamma + 2\kappa$, $\alpha_1 + 3\gamma + \kappa$, $\alpha_2 + 3\gamma + \kappa < 0$, we have

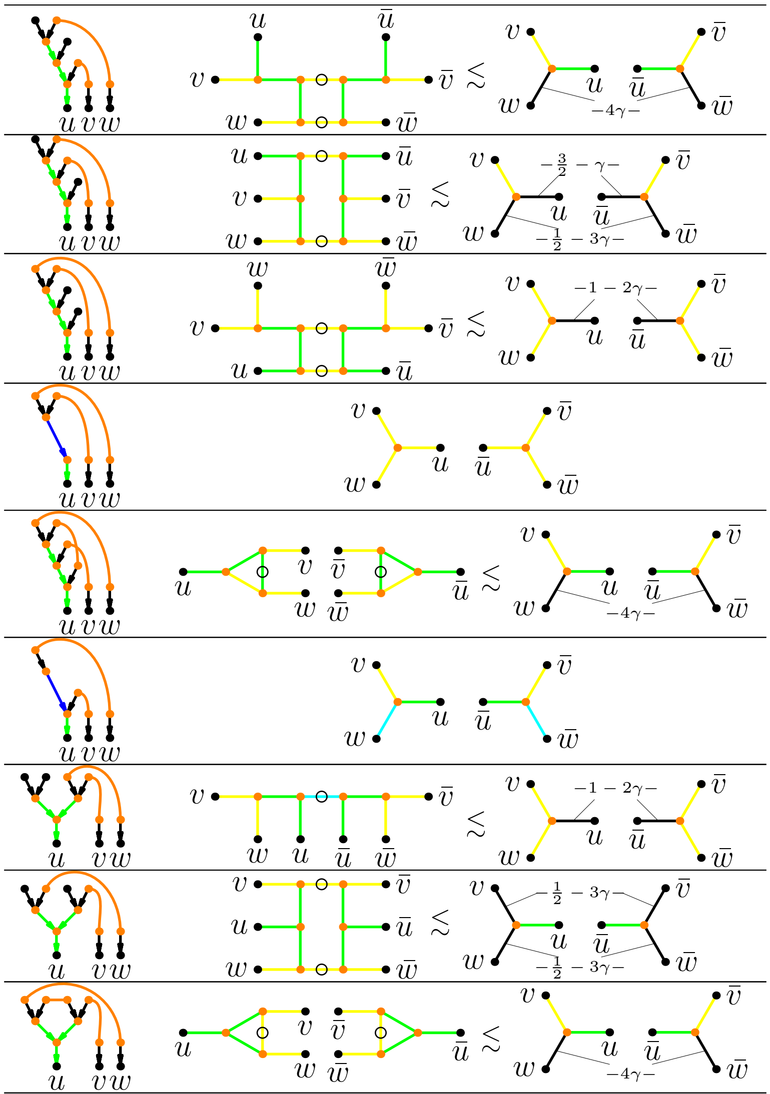


$$=: \|u - v\|_{\mathfrak{s}}^{-3\gamma-\kappa} \|u - w\|_{\mathfrak{s}}^{-3\gamma-\kappa} \sum_{i=1}^{4} I_i(u, v, w).$$

If $\alpha_1 + \alpha_2 + \beta + 6\gamma + 3 + 2\kappa > 0$, then we have $I_i \lesssim 1$ $(i = 1, \dots, 4)$.

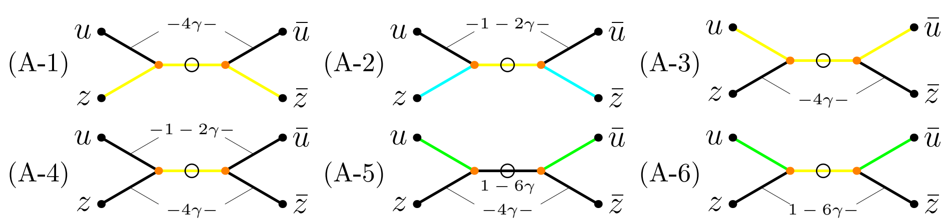
All contractions of graphs into (8.2) are shown in the following table:

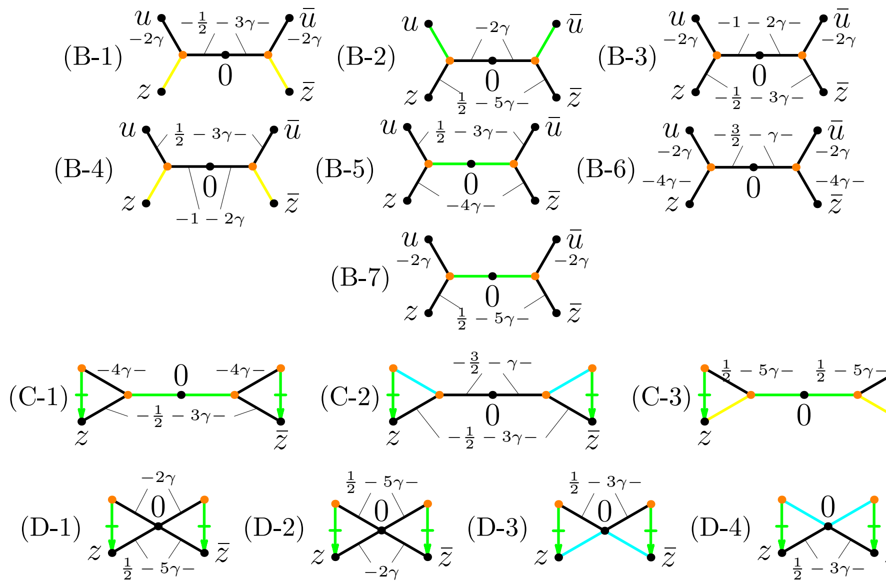
$\mathcal{W}(u, v, w)$	$(\mathcal{W}(u, v, w; \cdot), \mathcal{W}(\bar{u}, \bar{v}, \bar{w}; \cdot))_{L^2((\mathbb{R}\times\mathbb{T})^N)}$
	 $\lesssim$ 

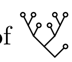


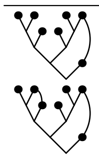
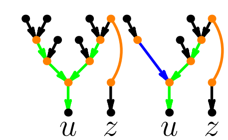
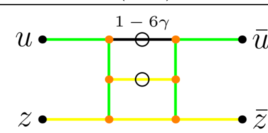
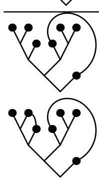
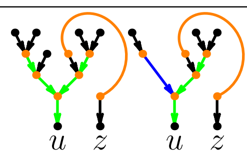
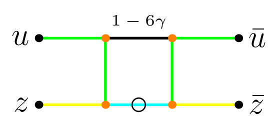
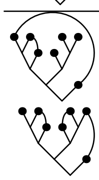
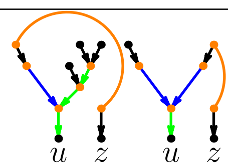
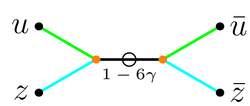
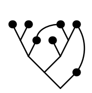
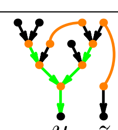
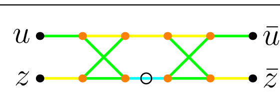
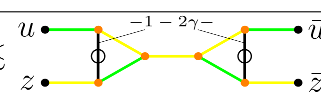
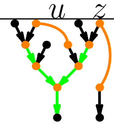
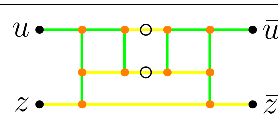
□

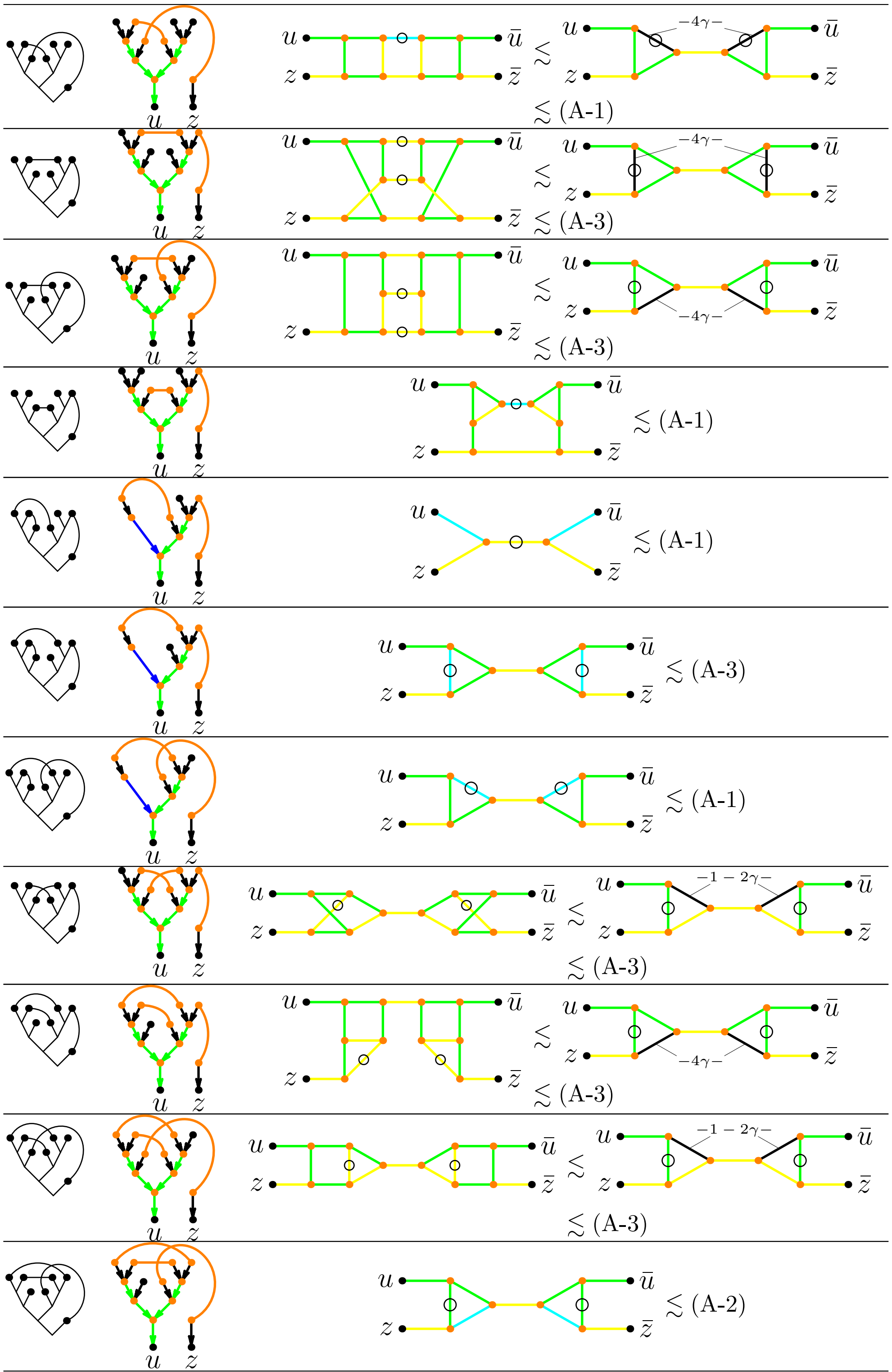
8.10. **Bounds of remaining terms.** It remains to estimates terms which are obtained one by one. Our goal is to obtain any of the following graphs.



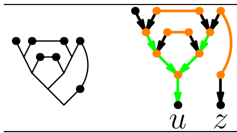
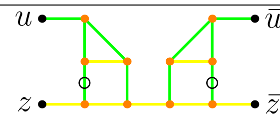


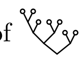
Bounds of 

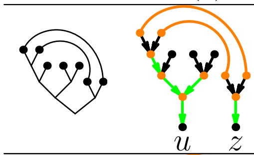
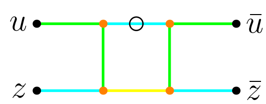
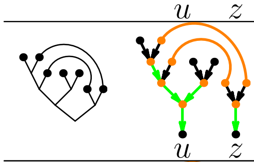
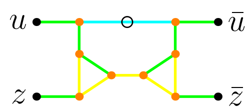
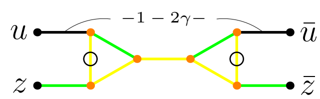
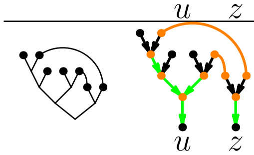
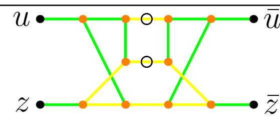
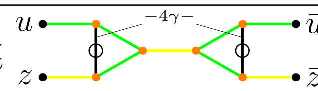
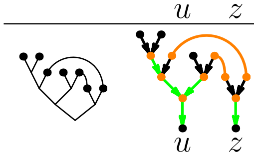
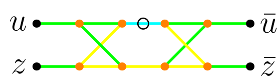
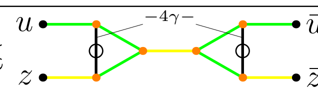
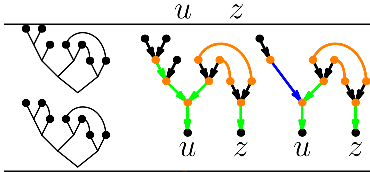
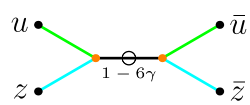
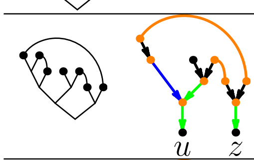
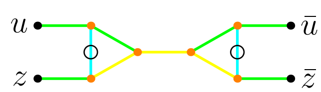
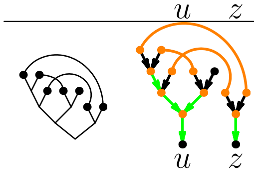
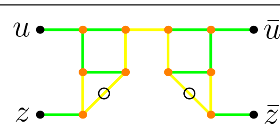
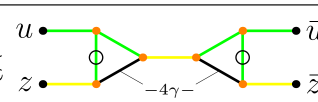
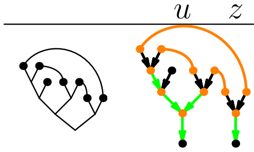
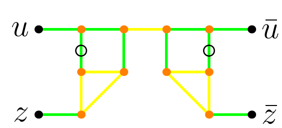
	$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$
		 $\lesssim$ (A-1)
		 $\lesssim$ (A-5)
		 $\lesssim$ (A-5)
		 $\lesssim$  $\lesssim$ (A-3)
		 $\lesssim$ (A-1)

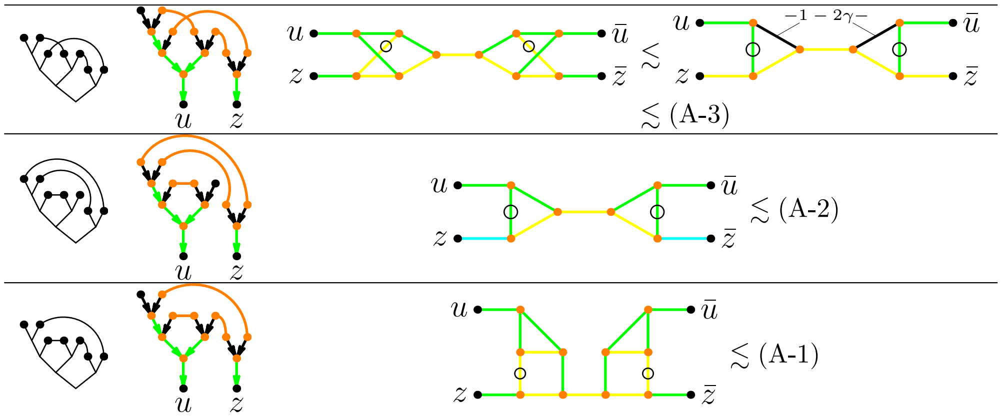


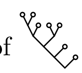
$\lesssim$ (A-1)
$\lesssim$ (A-3)
$\lesssim$ (A-3)
$\lesssim$ (A-1)
$\lesssim$ (A-1)
$\lesssim$ (A-3)
$\lesssim$ (A-1)
$\lesssim$ (A-3)
$\lesssim$ (A-3)
$\lesssim$ (A-3)
$\lesssim$ (A-2)

	 $\lesssim$ (A-1)

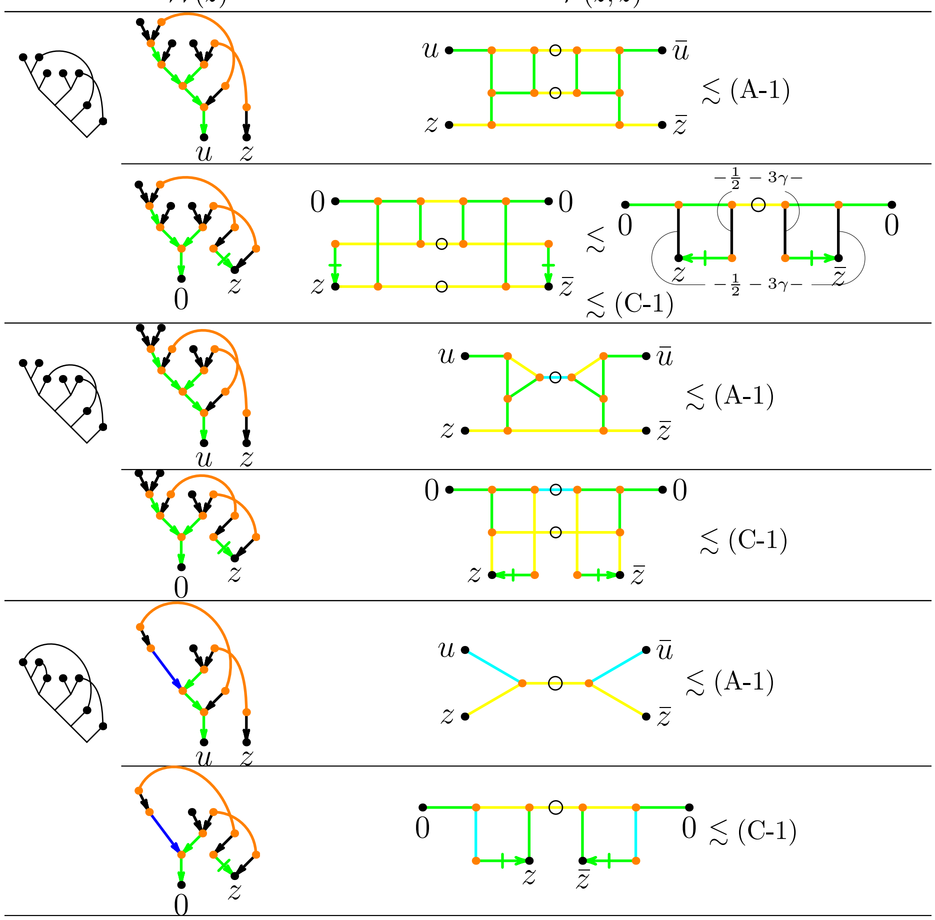
Bounds of 

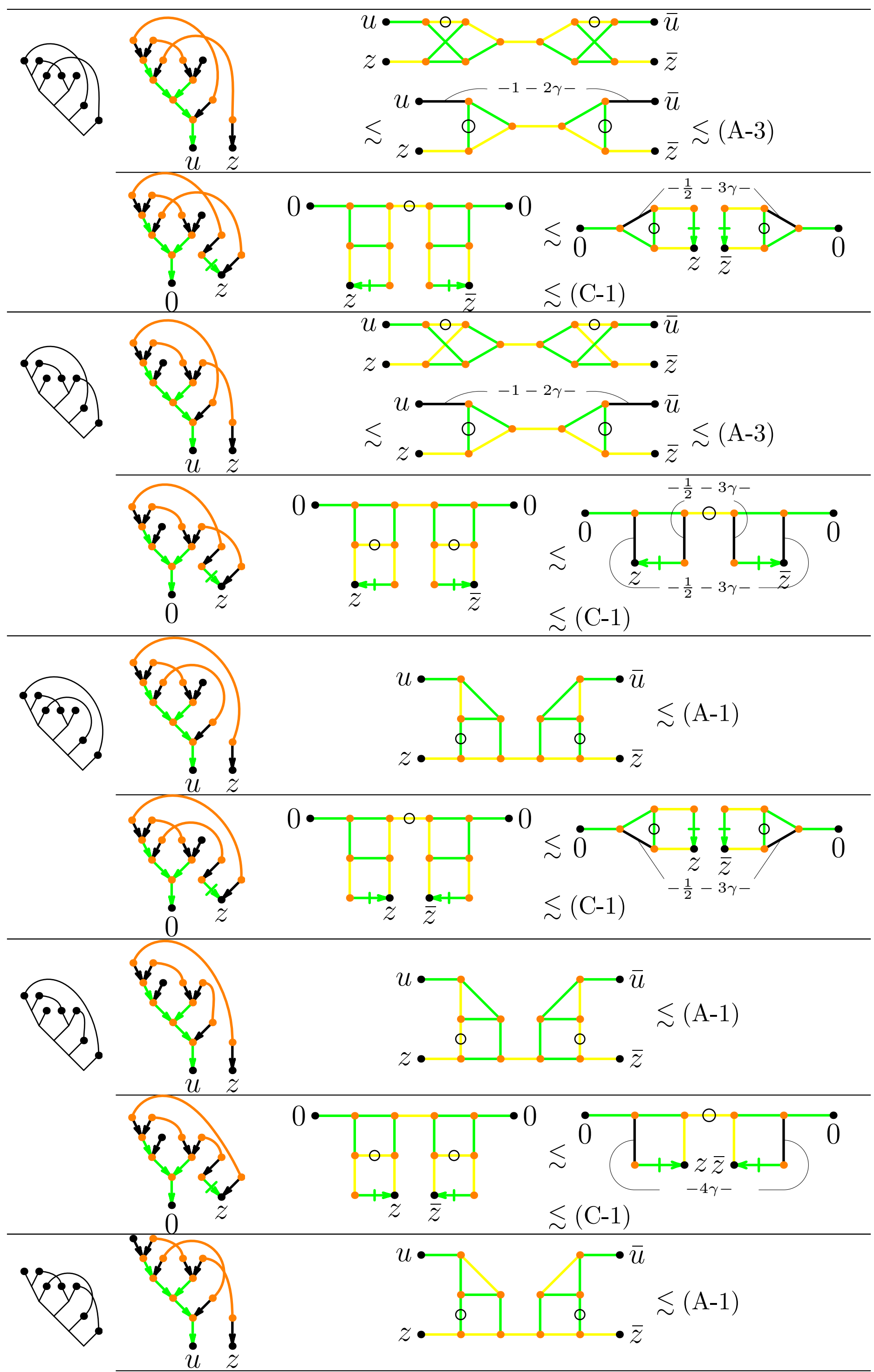
$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$
	 $\lesssim$ (A-2)
	 $\lesssim$  $\lesssim$ (A-1)
	 $\lesssim$  $\lesssim$ (A-3)
	 $\lesssim$  $\lesssim$ (A-3)
	 $\lesssim$ (A-5)
	 $\lesssim$ (A-3)
	 $\lesssim$  $\lesssim$ (A-3)
	 $\lesssim$ (A-2)



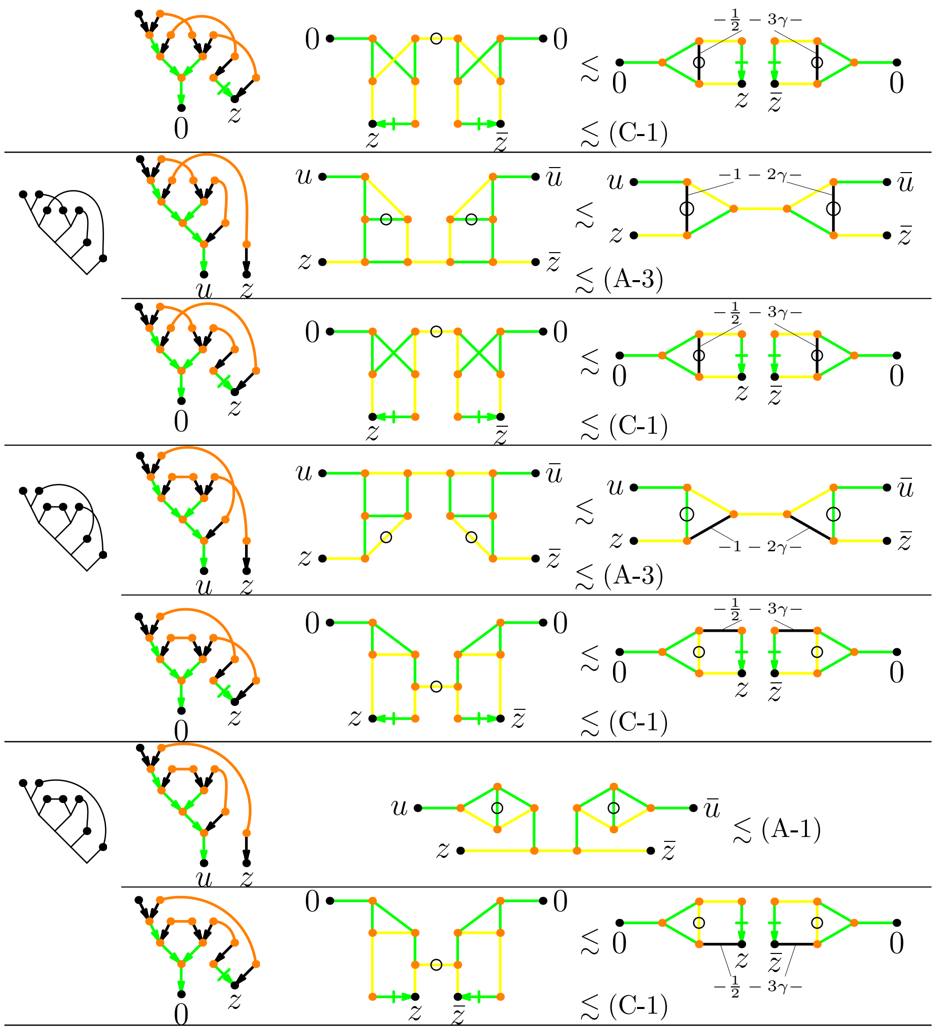
Bounds of  (the case (6.10))

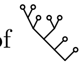
	$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$



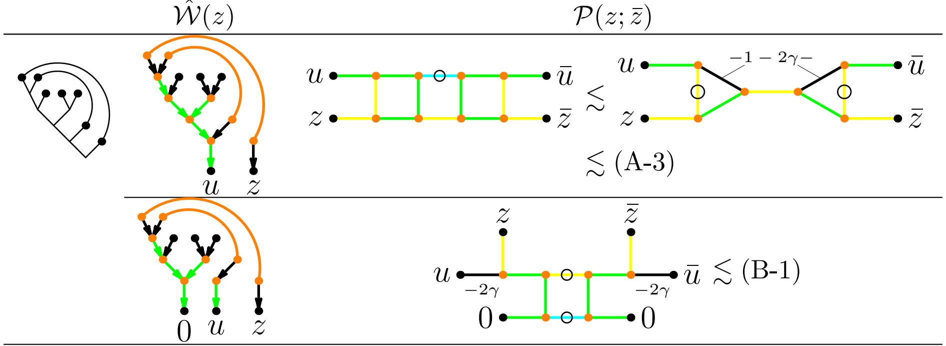


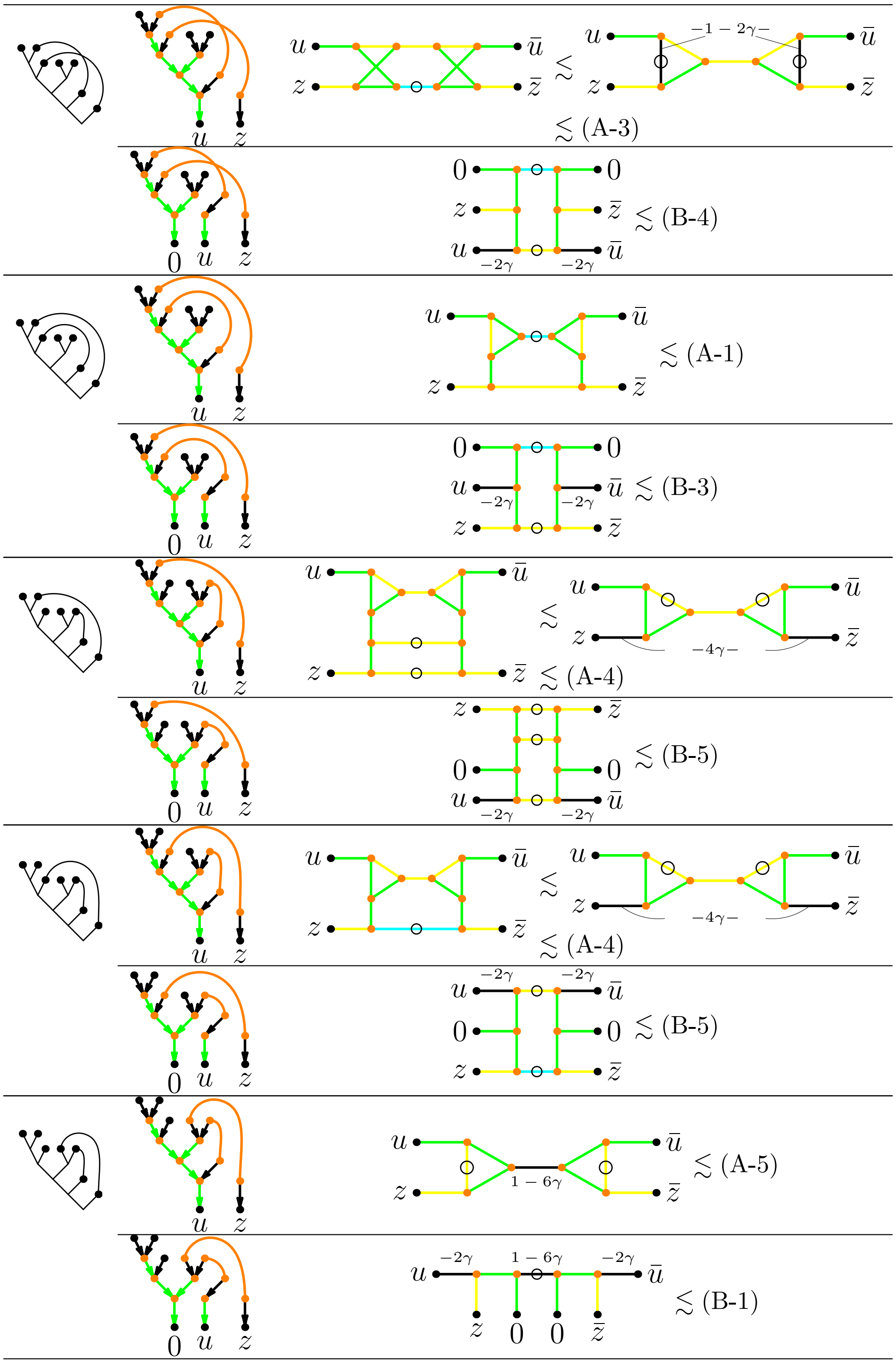
$-1-2\gamma-$
$\lesssim$ (A-3)
$-\frac{1}{2}-3\gamma-$
$\lesssim$ (C-1)
$-1-2\gamma-$
$\lesssim$ (A-3)
$-\frac{1}{2}-3\gamma-$
$-\frac{1}{2}-3\gamma-$
$\lesssim$ (C-1)
$\lesssim$ (A-1)
$-\frac{1}{2}-3\gamma-$
$\lesssim$ (C-1)
$\lesssim$ (A-1)
$-4\gamma-$
$\lesssim$ (C-1)
$\lesssim$ (A-1)

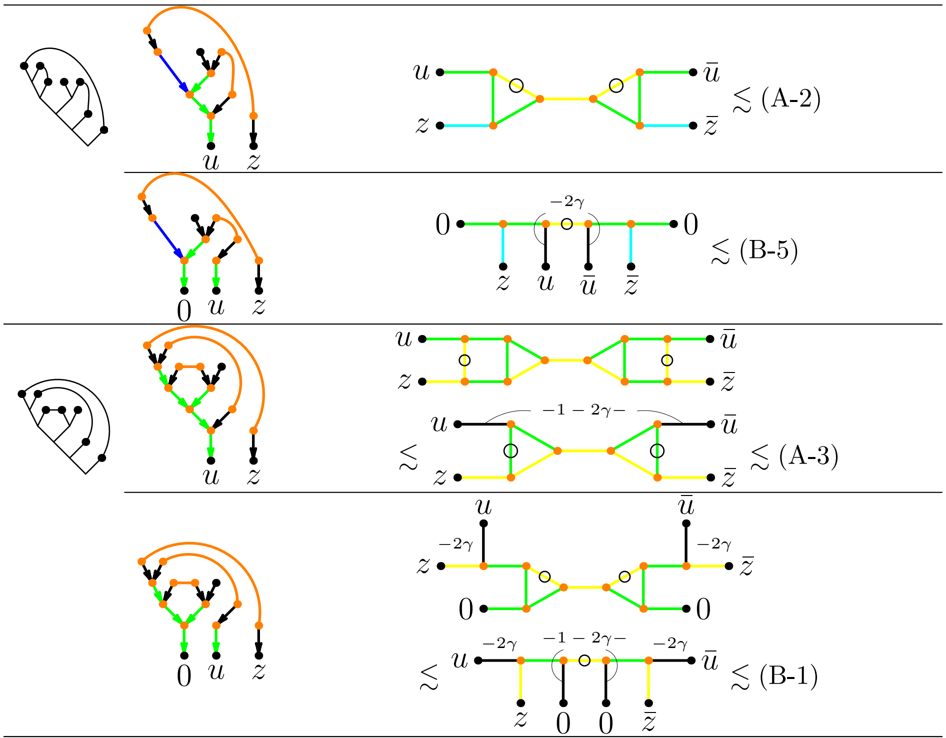


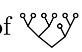
Bounds of  (the case (6.9))

$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$

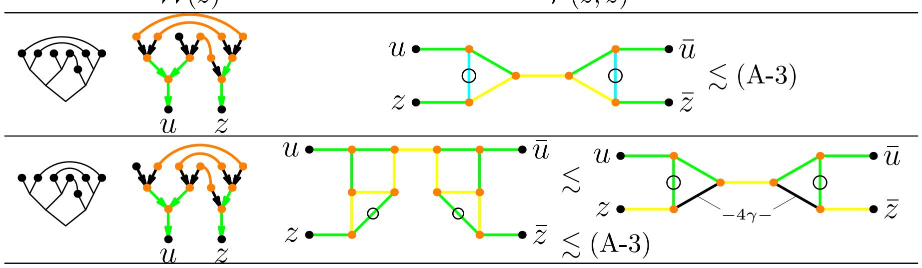


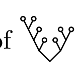

$-1-2\gamma-$
$\lesssim$ (A-3)
$\lesssim$ (B-4)
-2γ
$\lesssim$ (A-1)
$\lesssim$ (B-3)
$-4\gamma-$
$\lesssim$ (A-4)
$\lesssim$ (B-5)
$\lesssim$ (A-4)
$\lesssim$ (B-5)
$1-6\gamma$
$\lesssim$ (A-5)
$\lesssim$ (B-1)



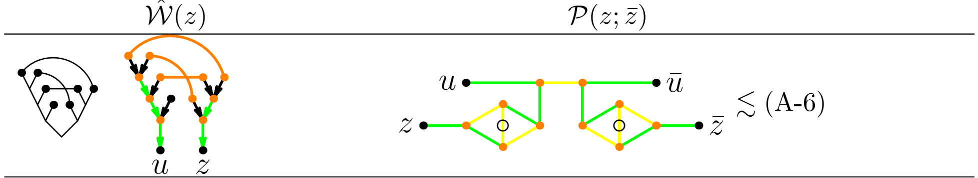
Bounds of 

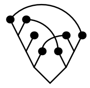
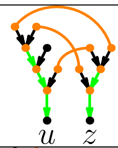
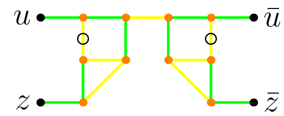
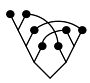
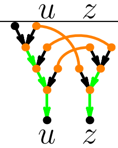
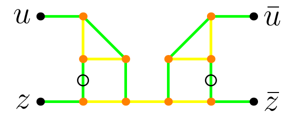
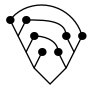
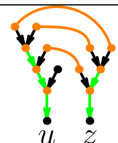
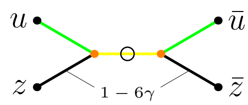
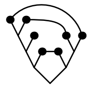
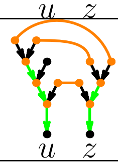
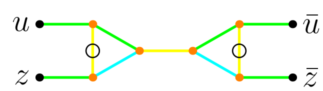
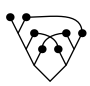
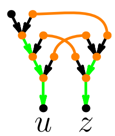
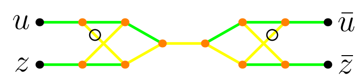
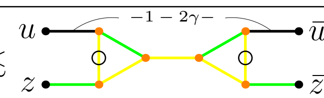
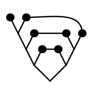
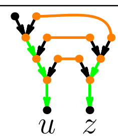
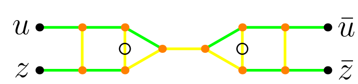
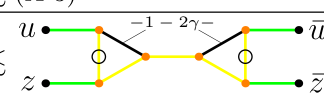
$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$

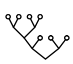
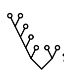


Bounds of 

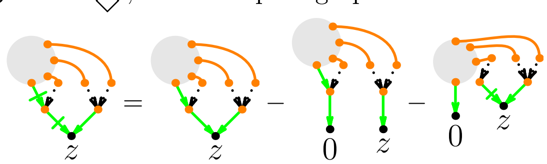
$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$



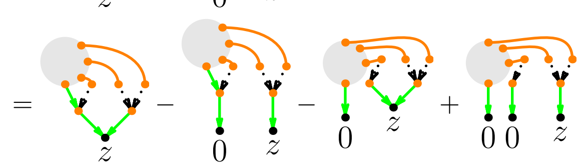
			$\lesssim$ (A-2)
			$\lesssim$ (A-1)
			$\lesssim$ (A-6)
			$\lesssim$ (A-2)
		 $\lesssim$ 	$\lesssim$ (A-3)
		 $\lesssim$ 	$\lesssim$ (A-3)

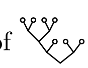
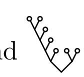
For  and , we decompose graphs as follows.

(8.3)

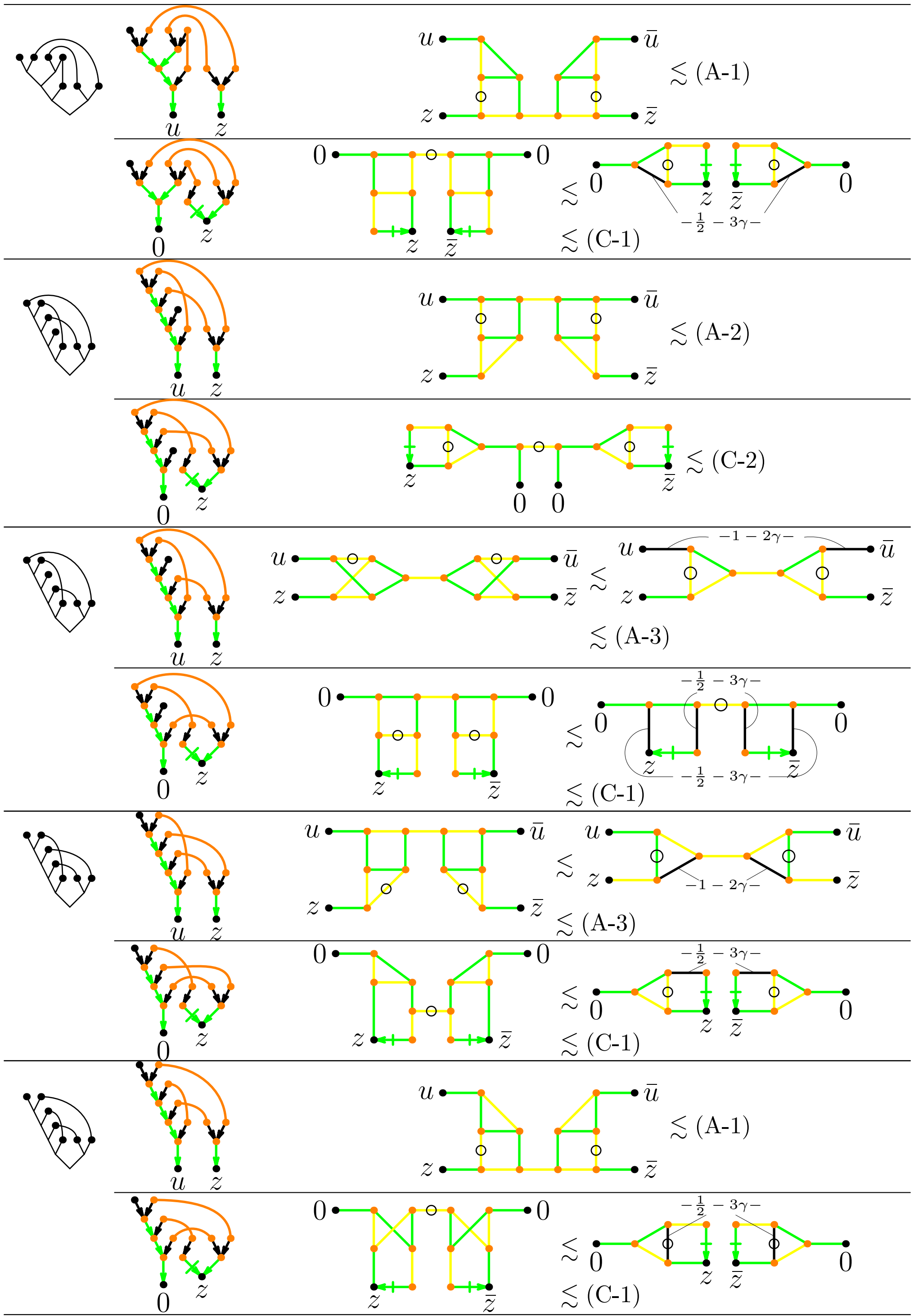


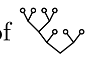
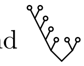
(8.4)

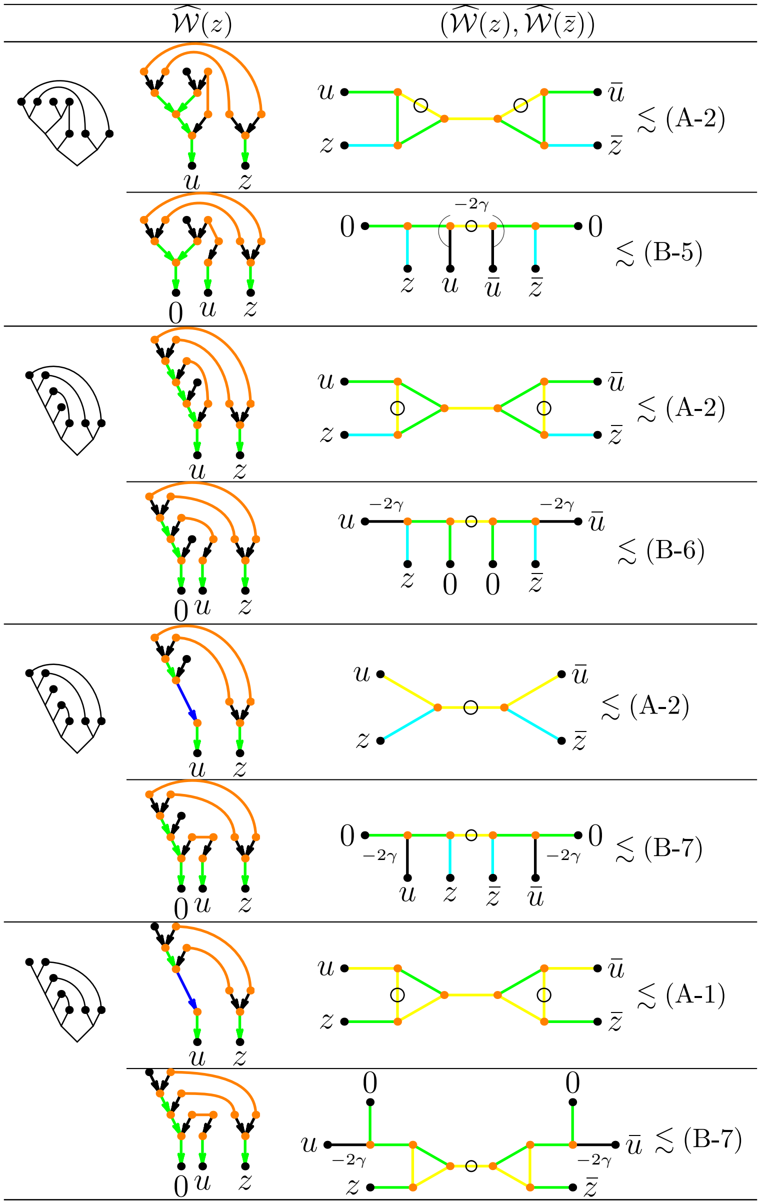


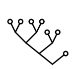
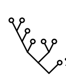
Bounds of  and  (the case (8.3))

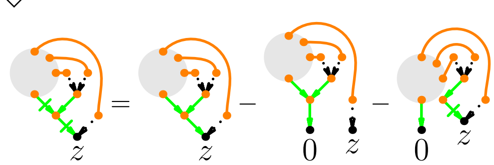
$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$

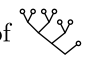
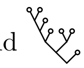

≲ (A-1)
≲ (C-1)
−1/2 − 3γ−
≲ (A-2)
≲ (C-2)
−1 − 2γ−
≲ (A-3)
−1/2 − 3γ−
≲ (C-1)
−1 − 2γ−
≲ (A-3)
−1/2 − 3γ−
≲ (C-1)
≲ (A-1)
−1/2 − 3γ−
≲ (C-1)

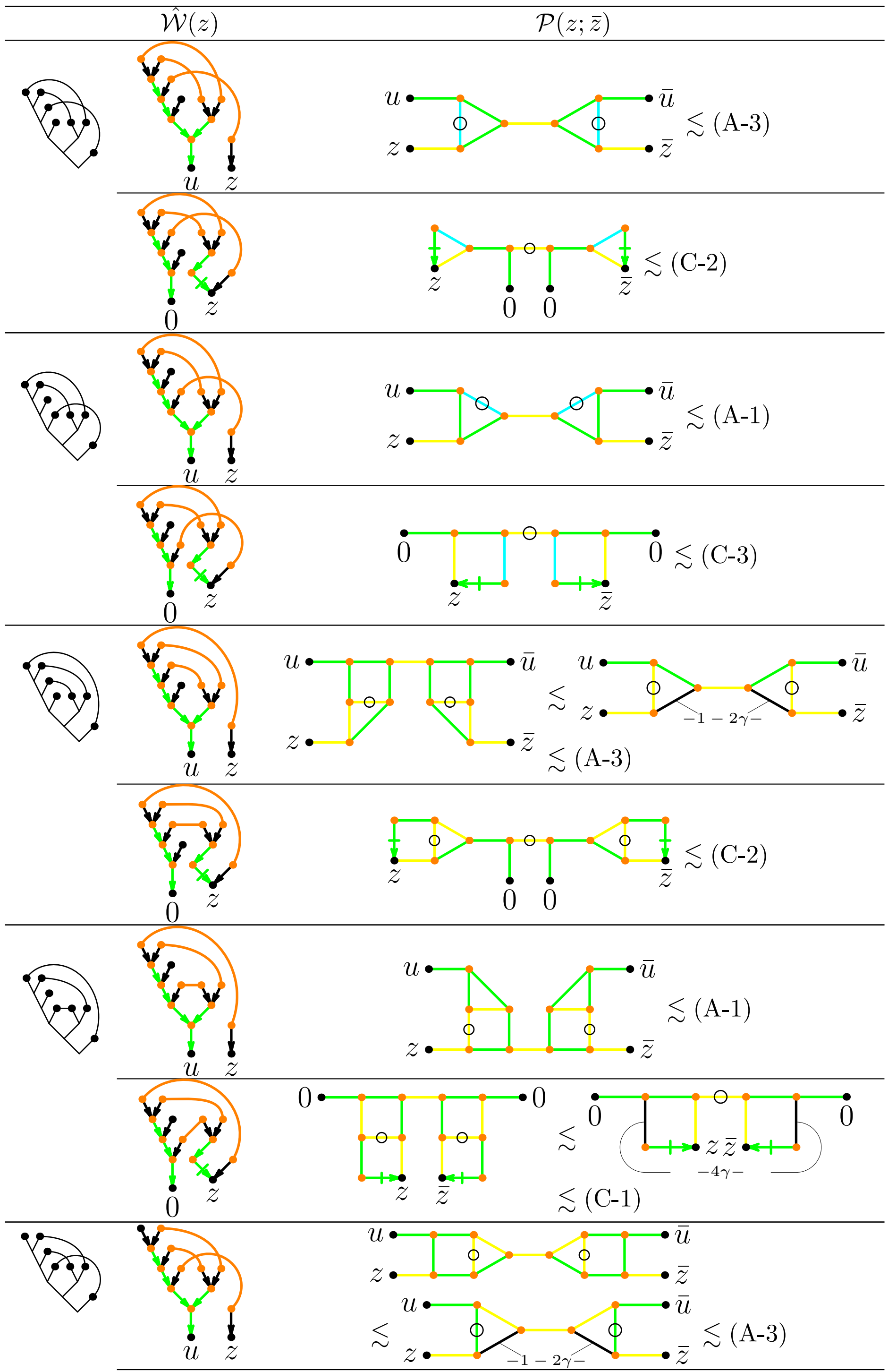
Bounds of  and  (the case (8.4))

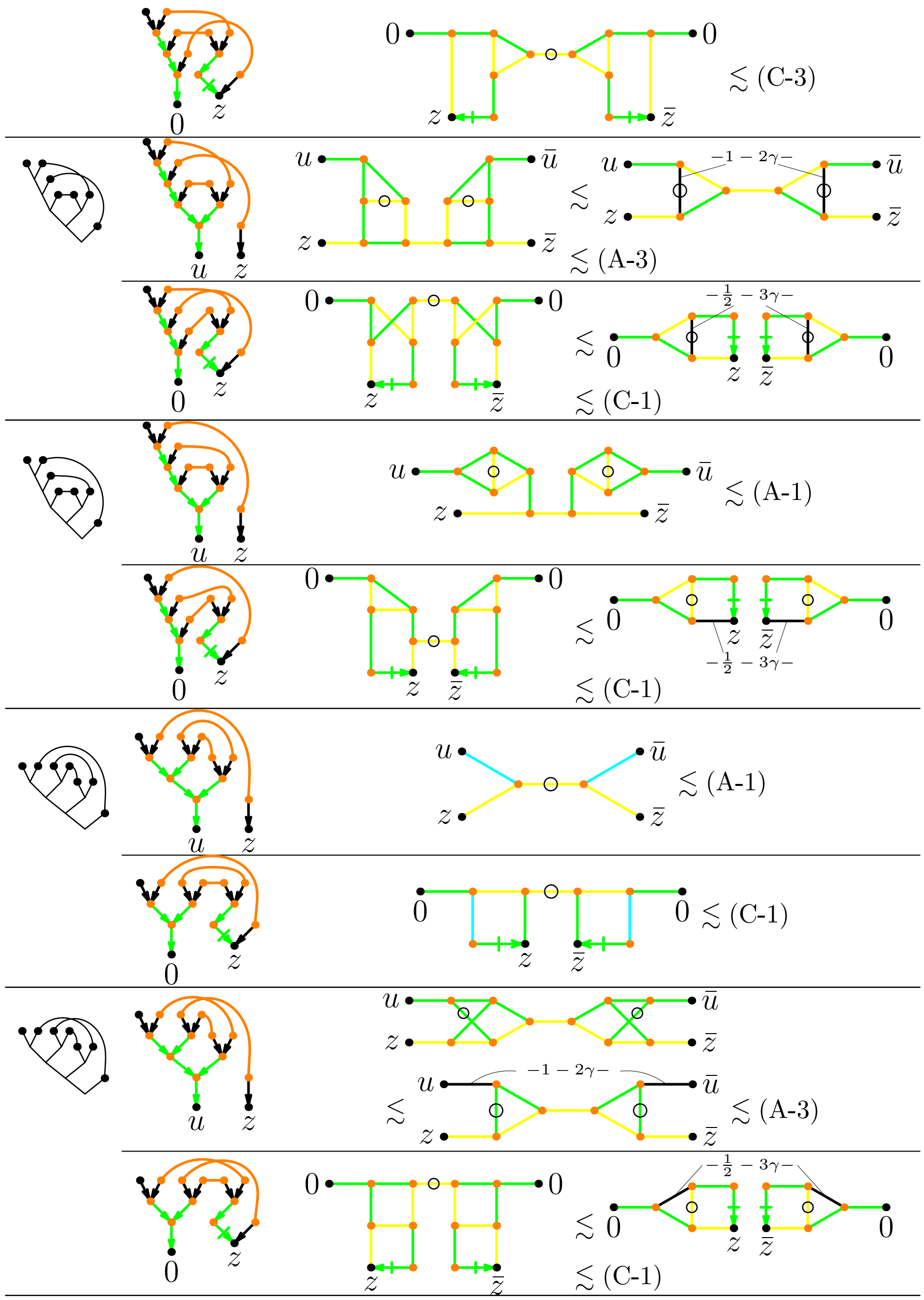


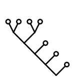
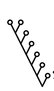
For  and , we decompose graphs as follows.

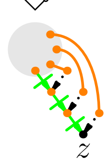


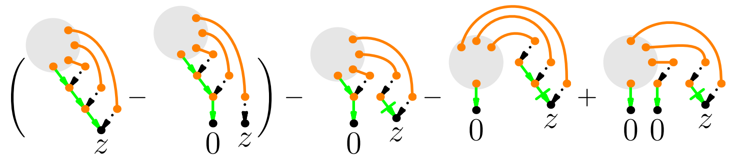
Bounds of  and 

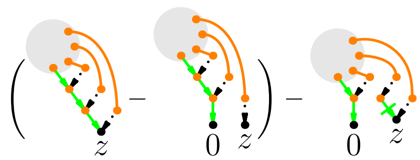


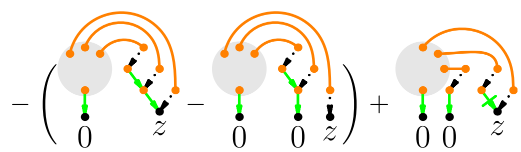


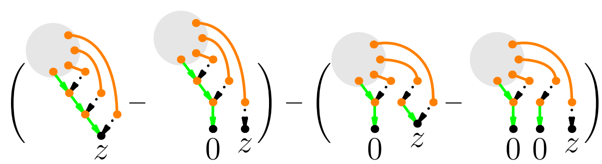
For  and , we decompose graphs as follows.

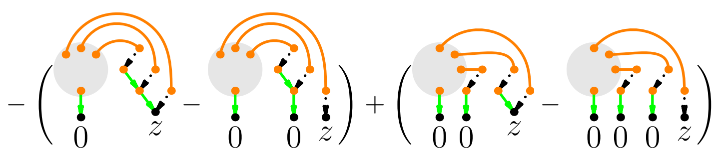


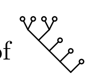
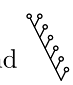
(8.5) $=$ 

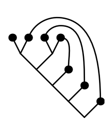
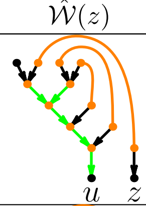
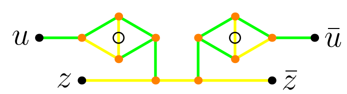
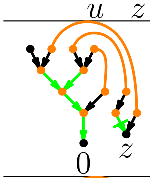
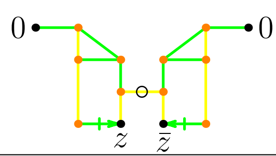
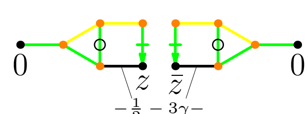
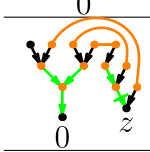
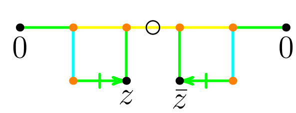
(8.6) $=$ 

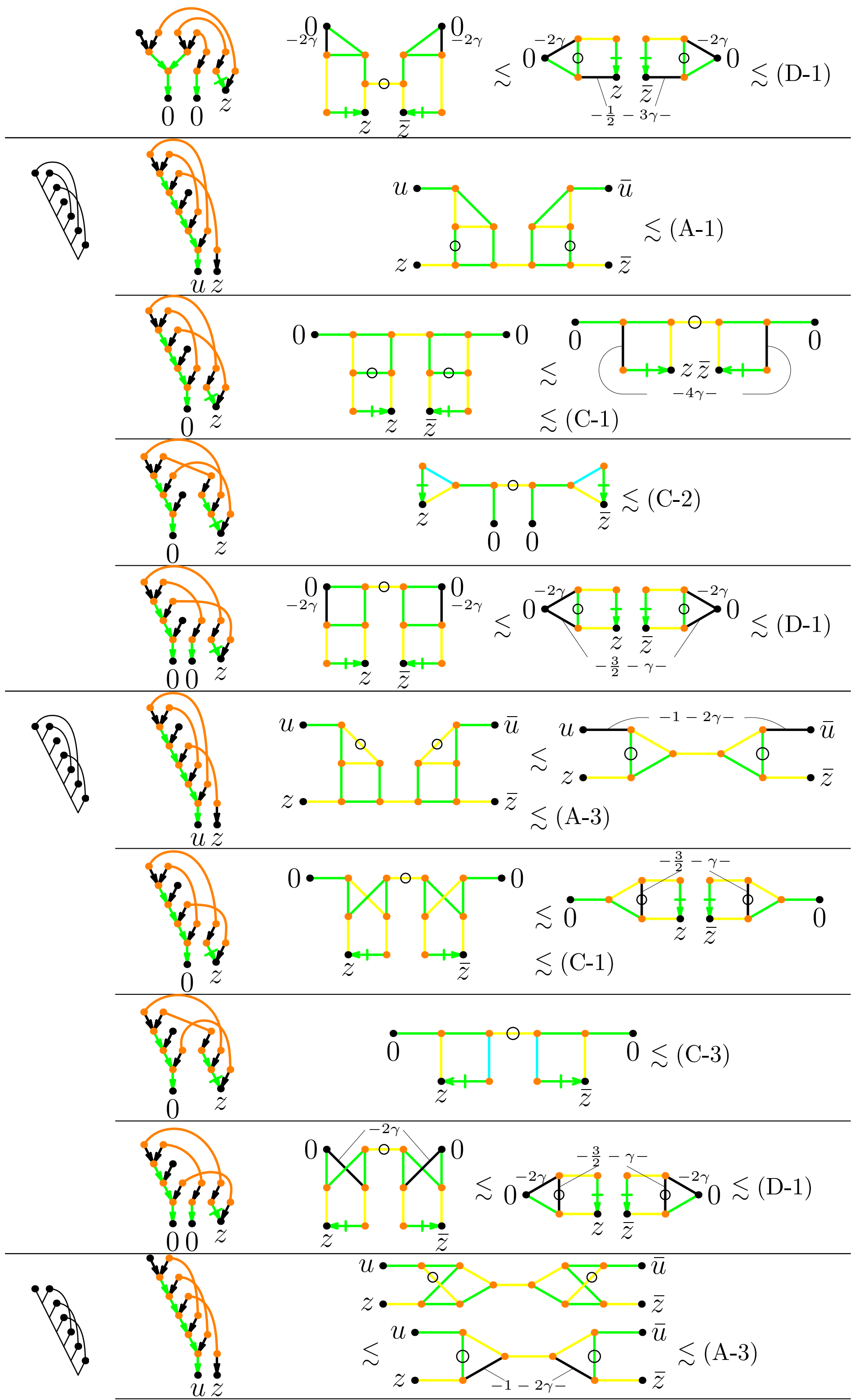


(8.7) $=$ 

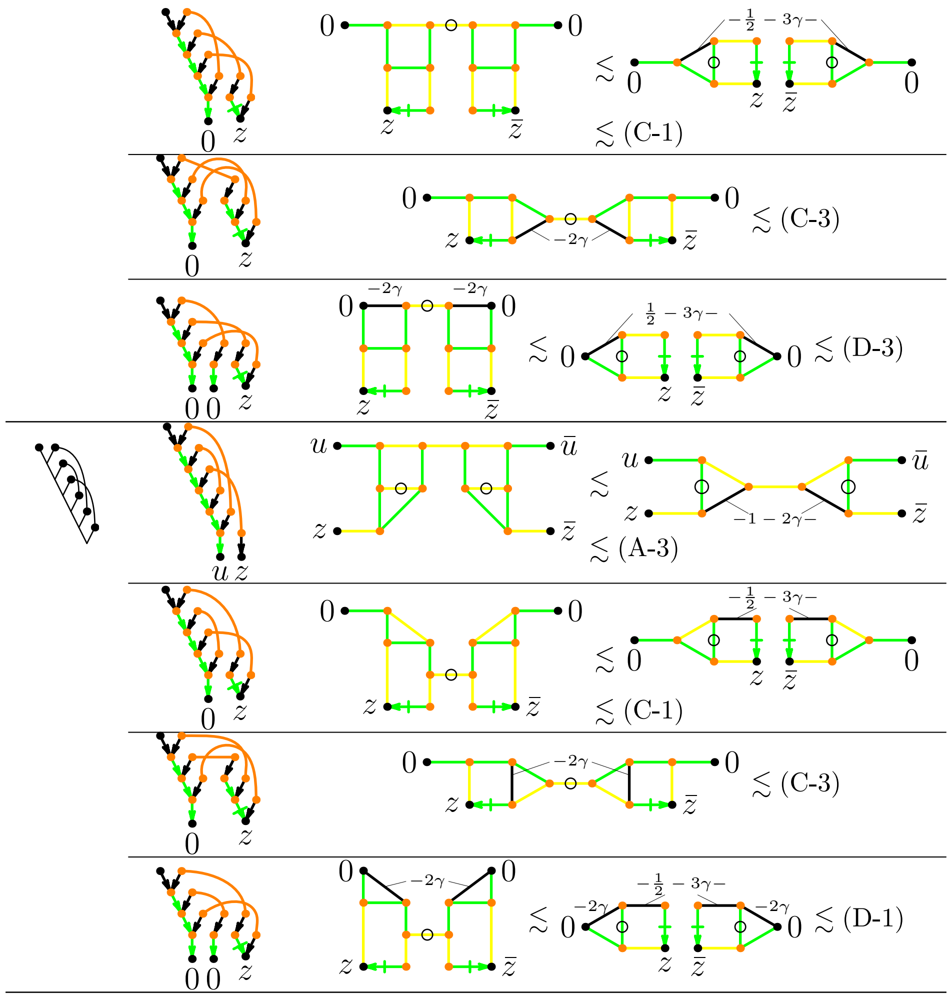


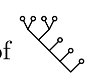
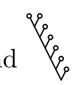
Bounds of  and  (the case (8.5))

	$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$
		 $\lesssim$ (A-1)
		 $\lesssim$  $\lesssim$ (C-1)
		 $\lesssim$ (C-1)

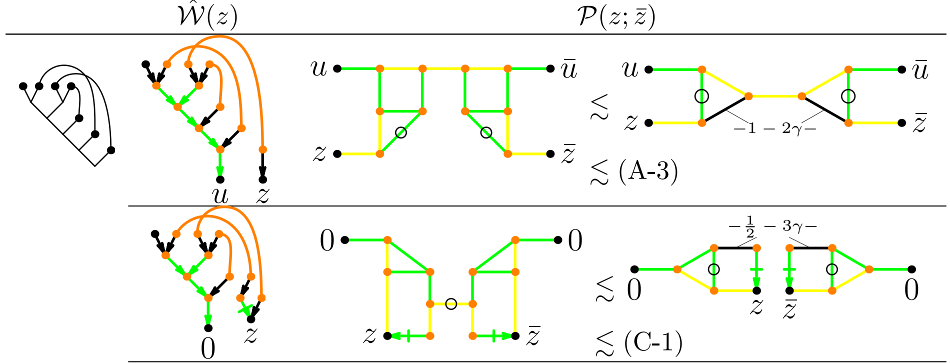


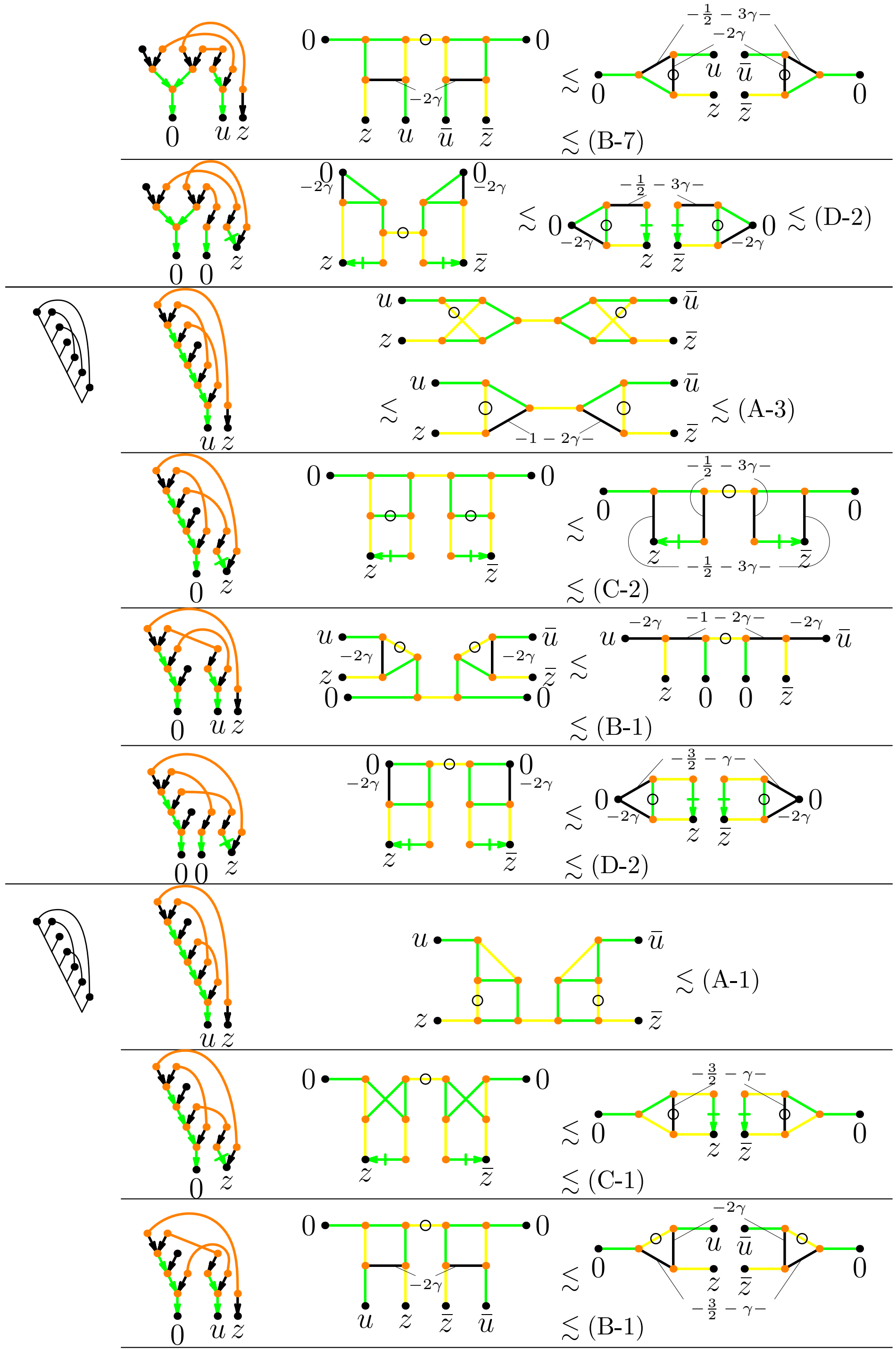
≲ (D-1)
$-\frac{1}{2} - 3\gamma-$
≲ (A-1)
$-4\gamma-$
≲ (C-1)
≲ (C-2)
$-\frac{3}{2} - \gamma-$
≲ (D-1)
$-1 - 2\gamma-$
≲ (A-3)
$-\frac{3}{2} - \gamma-$
≲ (C-1)
≲ (C-3)
-2γ
$-\frac{3}{2} - \gamma-$
≲ (D-1)
$-1 - 2\gamma-$
≲ (A-3)

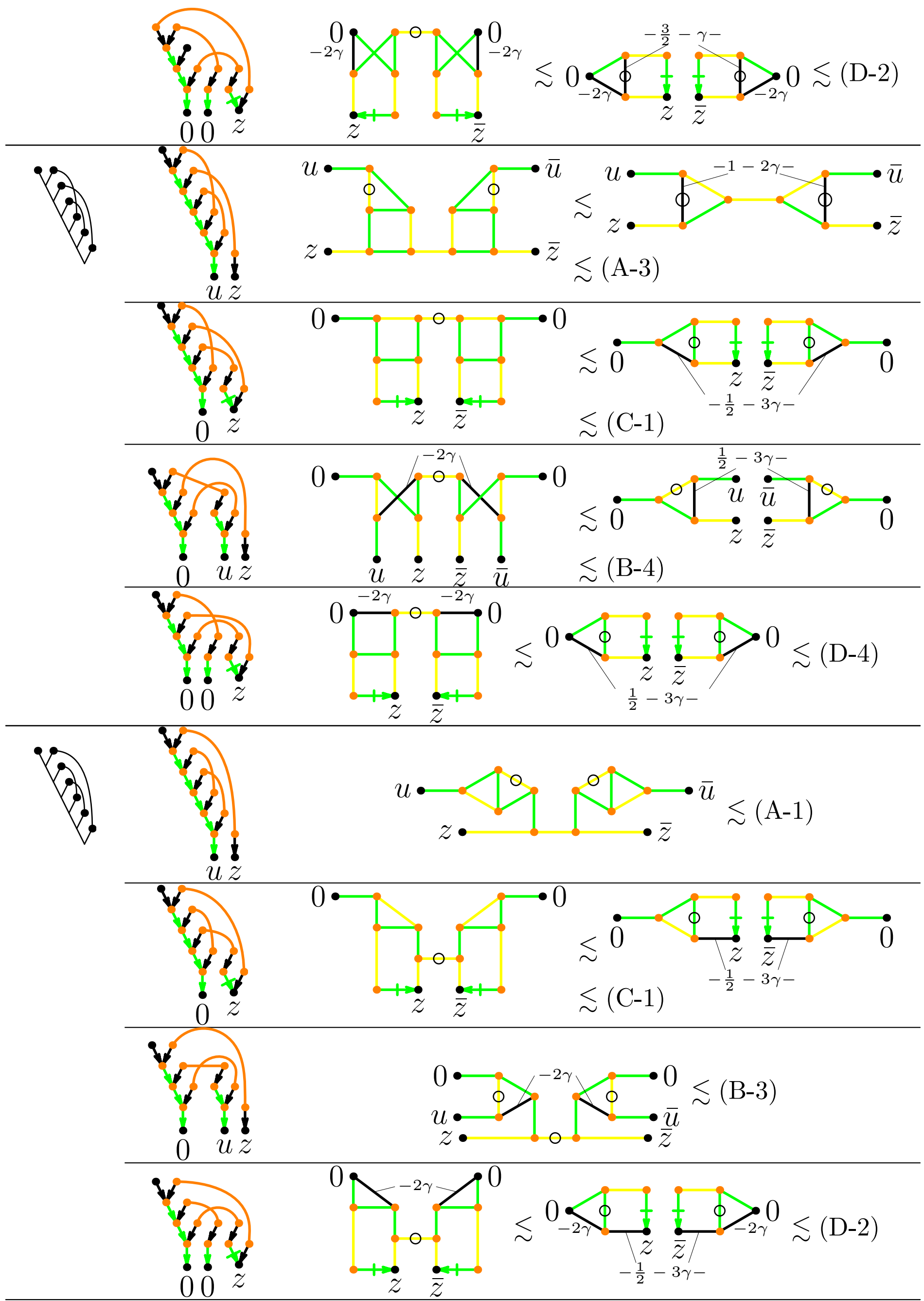


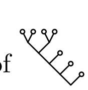
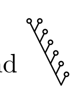
Bounds of  and  (the case (8.6))

$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$

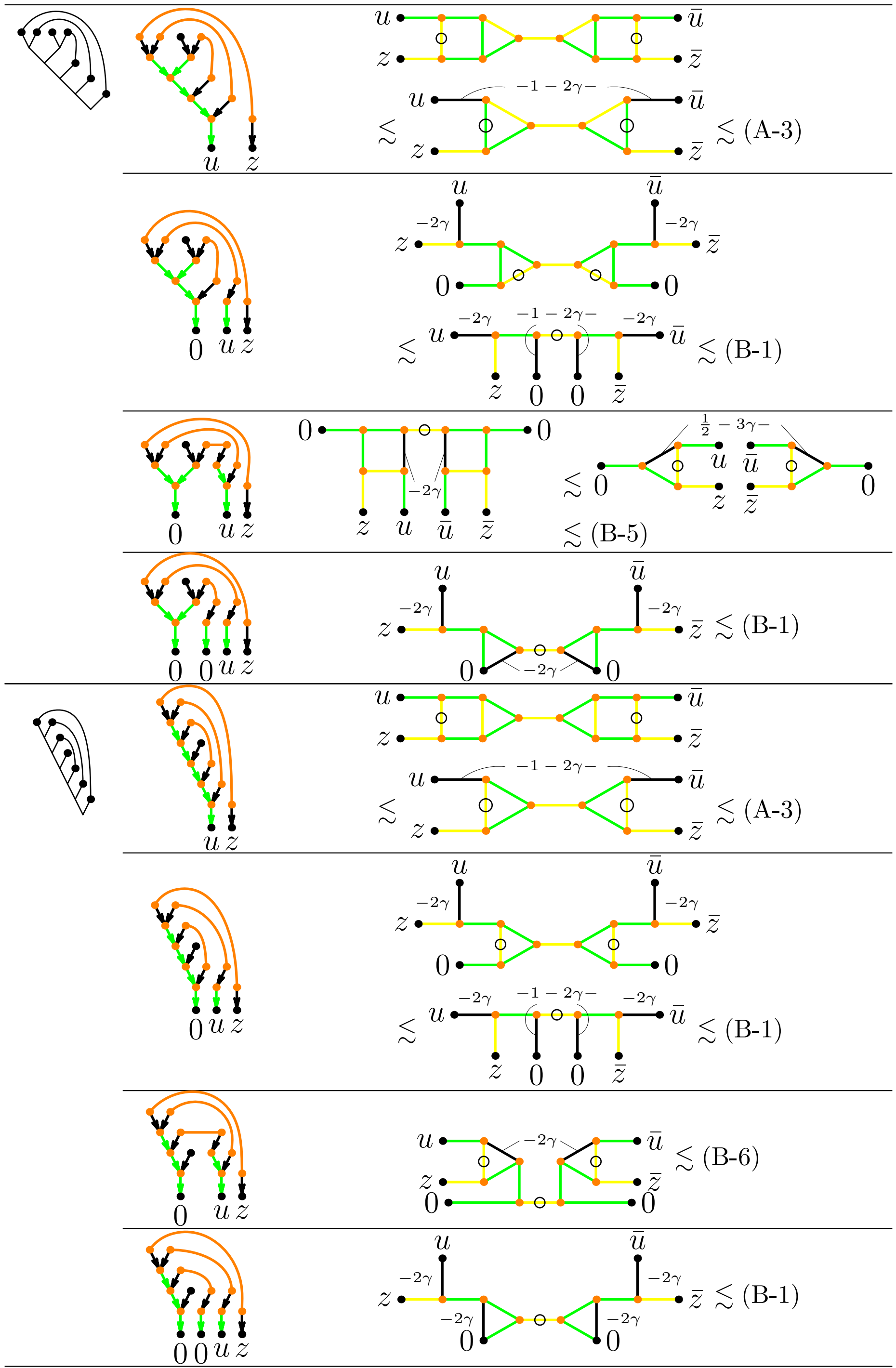


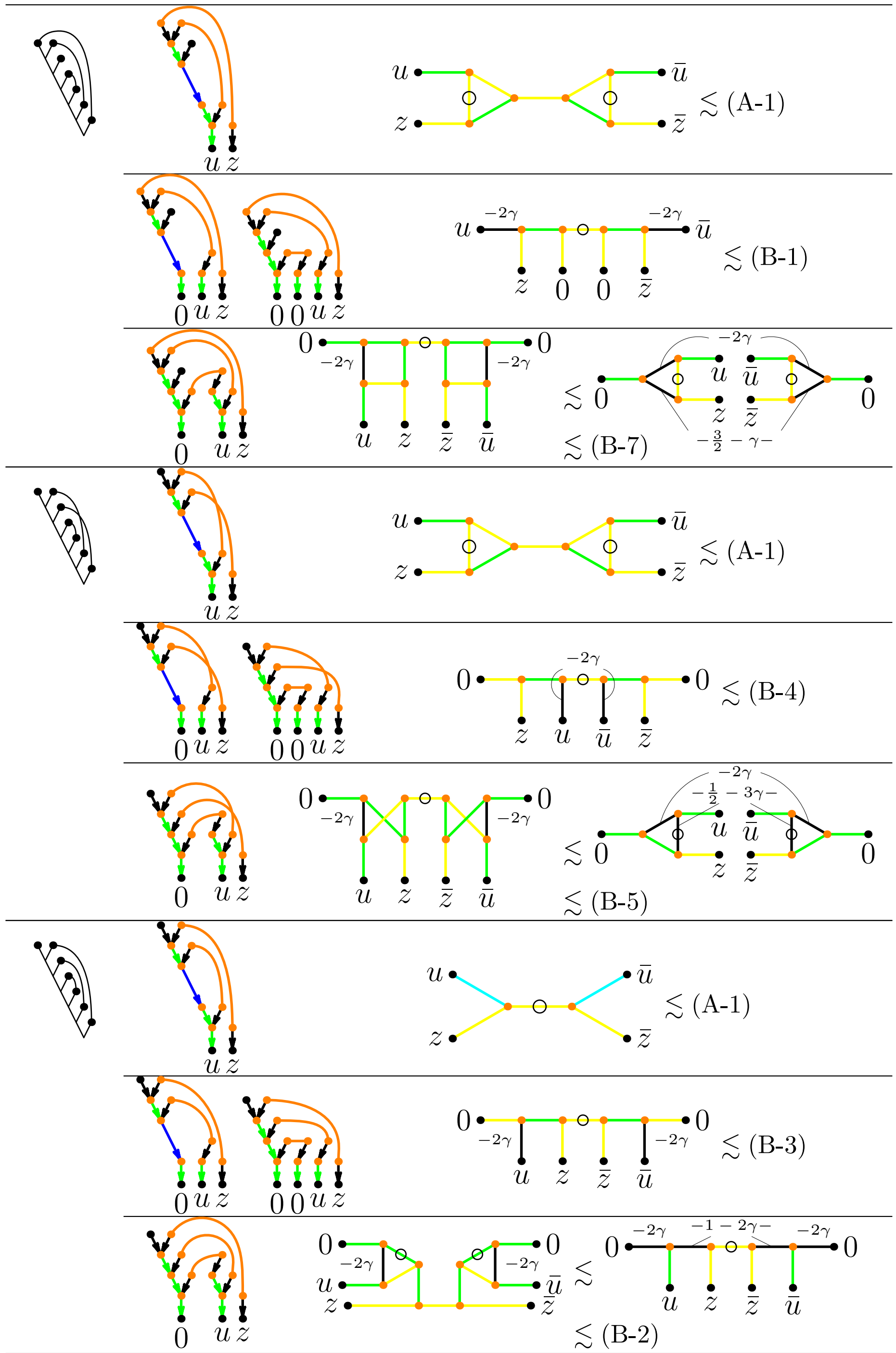

≲ (B-7)
≲ (D-2)
≲ (A-3)
≲ (C-2)
≲ (B-1)
≲ (D-2)
≲ (A-1)
≲ (C-1)
≲ (B-1)



Bounds of  and  (the case (8.7))

$\hat{\mathcal{W}}(z)$	$\mathcal{P}(z;\bar{z})$


≲ (A-3)
≲ (B-1)
≲ (B-5)
≲ (B-1)
≲ (A-3)
≲ (B-1)
≲ (B-6)
≲ (B-1)



≲ (A-1)
≲ (B-1)
≲ (B-7)
≲ (A-1)
≲ (B-4)
≲ (B-5)
≲ (A-1)
≲ (B-3)
≲ (B-2)

Acknowledgements

The author would like to thank Professor T. Funaki for leading him to the problem discussed in the present paper and asking Professor M. Hairer about this problem, who kindly pointed out that $\gamma = \frac{1}{4}$ is the border.

References

[1] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhikers guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012) 521-573.
[2] M. Hairer, *Solving the KPZ equation*, Ann. Math. (2) **178** (2013) 559–664.
[3] M. Hairer, *A theory of regularity structures*, Invent. Math. **198** (2014) 269–504.
[4] M. Hairer, *Singular stochastic PDEs*, arXiv:1403.6353 (2014).
[5] Y. Hu, J. Huang, K. Lê, D. Nualart, and S. Tindel, *Stochastic heat equation with rough dependence in space*, arXiv:1505.04924 (2015).
[6] S. Janson, *Gaussian Hilbert Spaces*, Cambridge Univ. Press, 1997.
[7] M. Kardar, G. Parisi, and Y.-C Zhang, *Dynamic scaling of growing interfaces*, Phys. Rev. Lett. **56** (1986) 889–892.

The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan
E-mail address: hoshino@ms.u-tokyo.ac.jp